AHT: the orbit-counting gap and a checked repair

Scoped review of Agol–Hass–Thurston, *The Computational Complexity of Knot Genus and Spanning Area*. Reviewed source: arXiv math/0205057v2, 10 July 2004. Review: 19 November 2026; repair incorporated 20 November 2026.

The printed progress argument for polynomial-time orbit counting has explicit counterexamples. A two-pairing family contradicts Claim 3 and every constant-times-k halving bound for its stated potential. A separate 29-point example defeats the application of Claim 4 across complete cycles. These are defects in the proof of Theorem 12; they do not establish superpolynomial running time, incorrect unweighted orbit counts, or a false NP-completeness theorem.

A separately checked replacement is now available. A modified schedule gives a correct unweighted orbit counter with polynomial bit complexity. With the boundary correction below, it supplies the three component-counting calls in the NP verifier, conditional on its imported normal-surface and topological premises. It does not validate the unchanged printed schedule or the weighted extension. Complete proof; proof PDF; repair checks.

The first wave combined the owner's earlier Astra/Fable working audit with fresh source-only Sol 6.2 reviews, focused Astra 6.2 challenges, source inspection and exact computations. Two initial Fable 5.2 requests exhausted their allowances. A subsequent wave produced six completed scoped Sol 6.2 and Fable 5.2 responses on the replacement, followed by coordinator adjudication. These are model assessments and source checks, not formal certificates or human peer review.

The source is the 29-page arXiv v2, not a newly collated journal copy. The corresponding journal article is in *Transactions of the AMS* 358 (2006), 3821–3850. We could not retrieve the AMS PDF for complete collation. We make no claim that every edition has identical wording. The source's 2018 PDF date comes from a dynamic TeX date and is not its submission date. Frozen edition; journal record.

Findings and their limits

Item	Adjudicated finding	What remains open here
Claim 3 and the $5k$-cycle bound	False for complete printed cycles; exact family below	Polynomial time of the original schedule
Claim 4 in the final complexity proof	Its telescope crosses coordinate contractions without the needed endpoint equality	The narrower fixed-coordinate inequality is not refuted
Weighted transfer, Section 6 Case 2	Wrong remainder interval; a six-point example changes orbit weights	A full checked replacement for the weighted algorithm
Euler characteristic in Section 5	Missing boundary-edge contribution for the exterior	Other verifier details are not comprehensively certified
Downstream work	Specific imports identified; pants-distance work has its own restricted progress argument	No later main theorem is refuted
Replacement algorithm	Astra's modified schedule now has separate operations, complexity and NP-interface checks	Unchanged schedule and full weighted extension remain unverified here

The NP-completeness result concerns a knot in a **varying triangulated orientable three-manifold**, not NP-hardness in the fixed three-sphere. Hardness and NP membership are separate branches. The orbit-counting defect affects the latter branch's exhibited polynomial-time justification.

1. Why the progress estimate matters

AHT represents components of normal surfaces by orbits of pairings on $\{1, \dots, N\}$. Endpoints are binary integers; N can be exponential in input length. An $O(N)$ step bound is insufficient. Theorem 12 needs a polynomial in $k \log N$, where k is the number of pairings.

Its cycles delete identities, contract static intervals, trim overlapping reversing pairings, perform available periodic mergers, transmit by a maximal pairing, and truncate an exclusive terminal suffix. For inclusive pairing widths w_i, the proposed potential is

$$X = 4^k \prod_i w_i.$$

The reference endpoint Z remains fixed until k decreases. A side $[u, v]$ is Z-close when

$$2(v - u + 1) \geq Z - u + 1.$$

Claim 3 is used to obtain progress in five cycles. Claim 4 converts the disappearance of close sides into a halving of X. The final proof claims halving after $5k$ cycles. Neither use survives the following examples. Replacing the incidental bound of k close sides by $2k$ does not repair them. Source: Theorem 12, PDF pages 14–18; TeX lines 1112–1443.

2. A two-pairing counterexample to constant-cycle halving

Choose $D = 2^s$, $s \geq 6$, and $W \geq 8D$. Put $A = W - D$ and $N_0 = Z = 2W + D$. Start with

$$g : [W, 2W + D - 1] \to [W + 1, 2W + D], \qquad g(x) = x + 1,$$

$$h : [D + 1, W + D] \to [W + D + 1, 2W + D], \qquad h(x) = 2W + 2D + 1 - x.$$

The first contraction removes $[1, D]$, changing the active endpoint to $2W$ but retaining Z. The translation is the unique maximal pairing after trimming. Its first transmission moves h's range left by $D + 1$.

At the end of cycle j, for $1 \leq j \leq s$, the exact state is

$$E_j = \left\lfloor \frac{D + 1}{2^{j-1}} \right\rfloor, \qquad N_j = 2A - 2 + E_j,$$

$$g_j : [A, N_j - 1] \to [A + 1, N_j],$$

$$h_j : [1, N_j - A + 1] \to [A, N_j], \qquad h_j(x) = N_j + 1 - x.$$

For the recurrence, trim the reflection. Its new range starts at $A + \lceil E_j/2 \rceil$, strictly right of g_j's range start in the continuing prefix, so the same translation wins again. Transmission shifts the reflected range left by $\lceil E_j/2 \rceil$ and truncation removes exactly that suffix, replacing E_j by $\lfloor E_j/2 \rfloor$. Only one pairing preserves orientation, so no merger is possible. There is no new static gap in this prefix.

The four closeness margins, in order g-domain, g-range, h-domain, h-range, are

$$W - 4D + 2E_j - 5, \quad W - 4D + 2E_j - 4,$$
$$2E_j - 3D - 2, \quad W - 4D + 2E_j - 3.$$

Their signs are $(+, +, -, +)$. Thus $k = 2$ and three sides remain close for arbitrarily many full cycles. Strict closeness does not rescue the claim. At a cycle's end the reflection may have the wider range, but loses that advantage at the next trim. The proof's passage from one cycle to transmission of the previous maximal pairing misses this possibility.

Moreover,

$$X_0 = 16W(W + D), \qquad X_j = 16(A - 2 + E_j)(A - 1 + E_j).$$

Since $E_s = 2$,

$$\frac{X_s}{X_0} = \frac{(W-D)(W-D+1)}{W(W+D)} \geq \frac{49}{72} > \frac{1}{2}.$$

For any fixed C, choose $s > Ck = 2C$. Then X has not halved within Ck cycles. This symbolic family needs no empirical extrapolation. Its slow-progress phase is itself only logarithmic in D, so it supplies **no superpolynomial lower bound**.

The exact checker reproduces the recurrence on 12 small cases with independent expanded orbit counts, a 12-cycle concrete example, and prefixes of 20 and 1,000 cycles with large binary endpoints. Sol, without receiving the earlier audit, found a different parametrization of the trimming obstruction; its trace was also independently executed.

Initial disjointness of each domain from its range does not exclude the phenomenon: a two-block lift with one extra translation returns to this family in one cycle. That constructs interval systems only. We have not realized the witnesses as compatible normal coordinates in a triangulated three-manifold.

3. A 29-point obstruction to the Claim 4 application

Set $N = Z = 29$, with increasing pairings

$$g : [14, 24] \to [15, 25], \qquad h : [1, 13] \to [17, 29].$$

Initially $X = 16 \cdot 11 \cdot 13 = 2288$. The first cycle selects h and truncates $[26, 29]$, leaving $h : [1, 9] \to [17, 25]$ and $X = 1584$.

The next cycle contracts the now-static $[10, 13]$. The active endpoint becomes 21, while $Z = 29$. The pairings are

$$g : [10, 20] \to [11, 21], \qquad h : [1, 9] \to [13, 21].$$

The translation g transmits h's range left by three, then truncates to endpoint 18. Final widths are 8 and 9, and all four sides are non-close. Nevertheless,

$$X_{\text{final}} = 16 \cdot 8 \cdot 9 = 1152 > 1144 = X_{\text{initial}}/2.$$

The telescope identifies the endpoint after one truncation with the endpoint before the next. Here they are 25 and 21, because of contraction. The inequality for compatible truncations in a common coordinate system is not refuted.

The earlier working audit has a small error here: after the second transmission both pairings are periodic, but no merger is available. Their periods are 1 and 9 and their span overlap has width 9, below the required 10. No deferral of an available merger is needed. Sol independently supplied another contraction example, with final potential ratio $57/100$; it also passed exact checks.

4. Two further local corrections

Weighted transfer. Section 6 Case 2 folds a constant-weight block $[r_j, s_j]$ around a period $t = c - a$. After assigning $\lfloor w_j/t \rfloor$ complete copies, the printed remainder begins at $a + \lfloor w_j/t \rfloor t$. It should begin at $r_j + \lfloor w_j/t \rfloor t$.

For $g : [1, 4] \to [3, 6]$ of period 2, put weight 1 on points 3 through 6 and weight 0 on points 1 and 2. Each residue orbit has total weight 2. The printed remainder is $[5, 6]$, returning weights 3 and 3; the corrected remainder is empty, returning 2 and 2. This refutes the literal transfer formula's weight-preservation claim. It does not verify the full weighted algorithm or its complexity. Source: Section 6, TeX lines 1800–1816.

Boundary Euler characteristic. For normal triangle, quadrilateral and vertex counts T, Q, V, and b boundary edges in the induced cell structure, $3T + 4Q = 2E - b$, hence

$$\chi = V - \frac{T}{2} - Q - \frac{b}{2}.$$

The closed-surface expression omits the final term. A normal triangle in a tetrahedral ball is a disk with $V = 3, T = 1, Q = 0, b = 3$: the corrected value is 1, while the closed-surface expression gives $5/2$. This matters in a knot exterior. Source: Section 5, TeX lines 1696–1706.

5. Which later arguments depend on this?

These are checks of specified passages, not entire later papers. A scalar component count, weighted component data and an explicit list of every component are different interfaces.

- **AHT's NP verifier** uses unweighted counts for a surface, its boundary and its doubled normal vector. A polynomial substitute for these calls suffices; the full weighted extension is not required. Bounded coordinate bit length does not make an $O(N)$ cycle bound adequate.
- **Lackenby's *The efficient certification of knottedness and Thurston norm***, Theorem 9.1, imports weighted AHT. Sections 9.4 and 13 use it for compressed cutting, bundle information and verifier tests. They contain substantial independent geometry but no replacement for the general AHT cycle bound. Author manuscript.
- **Lackenby–Yazdi's knot-genus manuscript**, Theorem 8.6, imports the preceding work's Theorem 9.3. This identifies a proof dependency, not a false main conclusion. Author manuscript.
- **Lackenby's survey** repeats the $5k$-cycle estimate and refers back to AHT. The classical-link paper also calls for component and Euler-characteristic data; its certificate-size bound does not itself compute them. Survey; link paper.
- **Lackenby–Yazdi's pants-distance work** is an exception to a blanket propagation claim. Its relevant track systems exclude mergers and trimming; Proposition 4.13 and Lemma 4.17 provide their own potential-drop argument. The general reflection obstruction does not refute that calculation. This checks the dependency, not all its geometric inputs. Author manuscript.

An exponentially large output cannot be explicitly listed in polynomial time merely because the input uses binary multiplicities. Compressed type/multiplicity lists and marked queries can avoid that obstruction. AHT's interval-list representation permits a compressed reading; we do not treat every use of "list" as a false theorem. Each application must retain the information it needs.

The inspected later passages do not supply a general replacement for the affected cycle bound. **This is not an exhaustive "never repaired" claim.** Schaefer–Sedgwick–Štefankovič explicitly state an alternative polynomial normal-surface component counter, but omit its surface proof in the inspected conference paper. It is a relevant possible alternative, not a proof checked here. Erickson–Nayyeri's tracing method treats curves on surfaces, not arbitrary weighted interval systems. HLP's earlier primitive-vertex certificate for unknot recognition is a separate special case. SSS; Erickson–Nayyeri; HLP.

6. The checked replacement and corrections to the audit

Astra's proposal changes the schedule: leave interior static points in place, hold a maximal pairing fixed while stabilizing transmissions and immediate trims, then truncate. The clean reconstruction proves orbit preservation, bounds each stabilization by an excess-halving recurrence, and gives an explicit potential argument between stabilized steps. All arithmetic is on integers of polynomial bit length. The proof supports arbitrary signed interval systems, including overlapping reflections; the original printed cycle order is not its subject.

Writing $B = 1 + \lceil \log_2(N+1) \rceil$, it bounds macrosteps by $O((k+1)^2 B)$ and transmissions/trims by $O((k+1)^3 B^2)$. Pairwise merger searches and arithmetic also have polynomial cost. Calling a stabilized iteration a macrostep does not hide its internal work: that work is separately bounded.

The NP-interface audit identifies three unweighted calls: one component for F, one for ∂F, and two for the normal surface with coordinates $2v(F)$. Ambient orientability must be promised or checked. The boundary-aware Euler characteristic is required, and the longitude argument uses integral null homology and the oriented surface, not parity alone. Minimum-genus fundamental witnesses, their coordinate bound and normal-surface realization remain imported premises.

Sol's different amortization proposal is still unaccepted. The adopted proof is the modified schedule, with the definitions and local derivations required by the separate audits. Full reconstruction.

The earlier audit also withdrew a concern that Schubert's cited existence result was limited to the three-sphere or excluded projective-plane summands. The original scan supports that withdrawal: Section 5.1 explicitly concerns compact three-manifolds; the ensuing cases discuss exceptional summands. A source-only model's uncertainty without that source is not an established defect. We have not checked the entire translation from Schubert's conventions to every modern normal-coordinate hypothesis. Original scan, printed pages 135–141.

Other topology and spanning-area objections in the returned assessments have not received sufficient independent checking for adoption here. They remain reviewer proposals. This release does not certify either whole proof branch.

7. Reproducibility and coverage

The reading view contains the frozen manuscript and a neutral argument map. The panel view contains this adjudication and a separate decision graph. A defective prerequisite blocks a proof path without refuting its consumer. The map indexes 23 source claims and the bibliography; only explicitly curated proof dependencies are classified. Graph endpoints are relative to this incomplete map.

The supplied 31-page working audit is retained privately with its hashes. Its corrected, checked portions are incorporated here; the uncorrected draft is not the public verdict. Sol's two initial requests received original TeX and all five figure pages without that audit. Astra's progress request received candidate objections and checker text; its dependency request received specified source excerpts. Neither had execution tools. The coordinator performed the exact runs and additional source checks separately.

Requests	Outcome
Sol 6.2 high, algorithm and topology	Two completed structured assessments
Fable 5.2 high, the same two packets	Both hit 65,536 output/reasoning tokens; one empty and one truncated answer; no completed verdicts
Astra 6.2 high, progress and dependency challenges	Two completed structured assessments
Sol 6.2 high, repair operations, complexity and NP interface	Three completed scoped responses
Fable 5.2 medium, the same three repair obligations	Three completed scoped responses

All six follow-up answers initially failed automatic ingestion because their source labels contained descriptive prose rather than exact packet identifiers. Two also declared missing background material. An explicit derivative reconciliation preserves the raw validation failures, partial source declarations and omitted-material limitations; it changes no mathematical field or verdict. Those six complete responses and their adjudication are not six natively successful validations. Readable records and reconciliation metadata accompany this release.

Conservative recorded usage totals **USD 6.363738 against the USD 25 ceiling**, including the two earlier incomplete attempts. The repair wave accounts for USD 1.0916055 of that total. These are usage estimates, not invoices. Reservations are not charges.

The original exact scripts cover both progress obstructions, independent Sol variants, the disjoint lift and local formula corrections. The separate replacement checker verifies the full equivalence relation after every move on 53,667 exhaustive small systems and 10,000 seeded random systems, plus large exact families. The finite checks support the implementation; the written proof supplies the universal bound. The original review records remain historical responses, including proposals and objections not adopted in this report.

Established within this audit: the examined text's progress proof fails; a changed schedule supplies polynomial unweighted orbit counting; with the displayed boundary correction and imported certificate premises, it repairs the audited NP-verifier interface. **Still outside that conclusion:** the weighted extension, complete verification of the topological foundations and hardness reduction, formal verification, author correspondence, exhaustive priority research, and complete journal collation. No later main theorem has been refuted.