Shortest-path improvement notes: scoped corrections

Math Referee: coordinator source checks

18 November 2026

This companion records corrections to two supplied AI-generated reconstructions of Benjamini–Tzalik, arXiv:2311.10014v1. It covers the planar and surface bounds, lower constructions and distance-four chain examined in the two-packet audit. It is not a complete review of either reconstruction, a formal verification, or a priority claim. Original documents are preserved.

1 A real degree budget in the Astra combined note

Theorem 1.1 of the supplied *Shortest paths: combined research note* allows $D \geq 4$ without explicitly requiring integrality. The rounded third term in equation (3) need not be bounded by the subsequent continuous square for real D. At $D = 9/2$ and $r = 1$ these expressions are, respectively,

$$\left(2 + \frac{\lfloor D/2 \rfloor - 2}{r}\right)\left(2 + \frac{\lceil D/2 \rceil - 2}{r}\right) = 6, \qquad \left(2 + \frac{D-4}{2r}\right)^2 = \frac{81}{16}.$$

This is an error in the intermediate comparison, not a counterexample to the theorem: the degree pair $(2, 3)$ is infeasible for $a + b \leq 9/2$.

To retain the real-parameter statement, replace the comparison on pages 5–6 by the following argument applied directly to feasible degrees $a, b \geq 2$:

$$\left(2 + \frac{a-2}{r}\right)\left(2 + \frac{b-2}{r}\right) \leq \left(2 + \frac{a+b-4}{2r}\right)^2 \leq \left(2 + \frac{D-4}{2r}\right)^2.$$

The first difference is $(a-b)^2/(4r^2)$. The cases $a = 1$ or $b = 1$ retain their existing bounds. Thus the final continuous upper bound and intended integer-degree applications are unchanged. The supplied Fable note's Theorem 8.4 instead makes the simpler explicit choice that D is an integer in the accompanying version 1.1.

2 Depth-one degree attainment in the Fable note

Proposition 8.5(b) should describe its graphs as having maximum degree *at most* $\max\{D, k_0\}$ and $\max\{D + 2, k_0\}$, respectively. For $h = 1$ no expansion occurs. The planar graph is K_{2,k_0} and its spherical completion is the bipyramid, whose maximum degrees are $\max\{2, k_0\}$ and $\max\{4, k_0\}$. For example, $D = 5, k_0 = 3$ gives degrees 3 and 4, rather than 5 and 7. Both root distances and the exact geodesic count $N = k_0$ remain correct.

For $h \geq 2$ an expanded rim vertex attains degree D, or $D + 2$ after adding the layer cycles. The correction has no effect on the distance-four equality examples or the asymptotic lower rate. It does not refute the existence of other graphs attaining the larger degree: suitable pendant or stellar-face insertions can achieve that without changing the root distance or count.

3 Small-degree detail at distance four

In Proposition 8.6, handle $D \leq 2$ before using the displayed estimates. For $D < 2$ an internal geodesic vertex is impossible. For $D = 2$ every such vertex has one parent and one child, so $N \leq \Delta$. The remaining proof therefore takes $D \geq 3$. Its upper bounds and equality thresholds are unchanged.

4 Attribution and evidence

The unrestricted multigraph and simple-graph exact maxima belong to Jing Yu and Jie-Xiang Zhu, *The Maximum Number of Shortest Paths in Graphs*, arXiv:2609.02334v1, Theorems 1.3 and 1.5. Their source was checked against the formulas in the supplied notes. The original degree-sensitive bound is due to Benjamini and Tzalik, arXiv:2311.10014v1, Theorem 1. The status of the additional planar claims in the wider literature remains unsettled; a targeted search is not evidence of novelty.

The accompanying source-adjudication report records all twelve checked obligations. Exact finite checks include 34,220 feasible degree-pair cases, 128 degree-repair graphs and 26 moment-matrix checks. These support the arguments and expose conventions; they do not replace general proofs.

The edited Fable note retains other sections verbatim with a front-page scope notice. Those sections and their computational or priority claims are not certified by this audit. The Astra PDF has no supplied editable source; this companion is its explicit correction, rather than a silent PDF edit.