NOTES ON "ON THE NUMBER OF SHORTEST PATHS IN GRAPHS" BY BENJAMINI AND TZALIK: CORRECTIONS, SHORTER PROOFS, AND EXTENSIONS

Abstract. We examine the note [2] of Benjamini and Tzalik. Its main theorem is correct, but several of the remarks that follow it need correction, and its two concrete open problems were settled in September 2026 by Yu and Zhu [8]. We record (i) three short proofs of the main theorem, one of which identifies the headline bound $2(\Delta/2)^t$ with the Carne–Varopoulos inequality at time $n = d(x, y)$; (ii) the corrected random-walk statement; (iii) an explicit classification of the extremal multigraphs for odd $\Delta \geq 5$; (iv) precise large-girth statements, including cubic extremal graphs of arbitrarily large girth; (v) a repaired and generalised version of the proposition on triangulated spheres; and (vi) for planar graphs, the exact maxima 2Δ and $\Delta(\Delta + 1)$ at distances three and four, and the fact that the exponential growth rate is $\sqrt{\Delta}\,(1 + o(1))$ rather than $\Delta/2$.

Version 1.1 scope. This derivative incorporates the checked integer-degree qualification in Theorem 8.4, the degree-budget correction in Proposition 8.5(b), and the small-degree detail in Proposition 8.6. Only the weighted planar/surface upper-bound chain, expanding constructions and distance-four argument received the scoped audit reported with this package. Other sections, computational claims and priority claims are retained from the supplied note and are not certified by that audit. The original source and a complete patch are preserved separately.
Provenance. These notes merge two independently produced machine analyses of [2] (A: by Claude; B: by GPT). In Table 2, "AB" marks items both analyses found independently, "A" and "B" items found by only one, and "merge" items that arose when combining them; every item was re-derived or numerically re-checked during the merge (Section 10). No exhaustive literature search was made; apart from Section 4, which reports results of [8], we do not know the statements below from the literature, but several (notably the spectral proof) belong to standard families of arguments.

1. Assessment

2. Setup and the two-walk inequalities

Throughout, G is a finite multigraph of maximum degree at most $\Delta \geq 2$, x, y are vertices at distance $t \geq 1$, a *geodesic* is a shortest x–y path recorded as its sequence of edges, $\mathcal{P}$ is the set of geodesics, $N = n_G(x, y) = |\mathcal{P}|$, and

$$q = \lfloor \Delta/2 \rfloor \lceil \Delta/2 \rceil .$$

Let H be the union of all geodesics and $V_i = \{v \in V(H) : d(x, v) = i\}$. Then $V_0 = \{x\}$, $V_t = \{y\}$, and every edge of H joins consecutive layers. For $v \in V_i$ let $d^-(v)$ and $d^+(v)$ be the numbers of edges of H (with multiplicity) from v to V_{i-1} and to V_{i+1}. For an internal vertex, $d^-(v) + d^+(v) = d_H(v) \leq \Delta$, hence $d^-(v)d^+(v) \leq q$.

For a geodesic P with vertices $x = v_0, v_1, \ldots, v_t = y$ put

$$F_P = \prod_{i=0}^{t-1} d^+(v_i), \qquad G_P = \prod_{i=1}^{t} d^-(v_i).$$

Date: 18 November 2026. Version 1.1: scoped corrections; not fully refereed.

Part of [2]	Verdict	See
Theorem 1	Correct. Index slips in the displays of the entropy proof; the Markov property it invokes is true but not needed.	§2
Conjecture 2, Question 3	Settled by Yu–Zhu [8].	§4
Simple graphs	Upper bound correct. The "extremal" example is miscounted by a factor 2 and is not extremal; for even Δ the stated upper bound is in fact sharp.	§4
Random-walk remark	False as stated, already for simple random walk on a 5-vertex graph. True when $\pi(x) = \pi(y)$; in general a factor $\sqrt{\pi(y)/\pi(x)}$ is needed.	§3
Large-girth remark	Needs precision: no saving at all for $\Delta = 3$; the sketched construction does not control the girth.	§6
Proposition 4 (sphere)	Statement correct, written proof defective. A short link argument proves more. For $\Delta \geq 7$ the bound has the wrong exponential rate.	§7, §8
Conjecture 5	The hypothesis "irreducible" is defined but never used. Open.	§9
Question 6	Ill-posed under the two literal readings.	§9

TABLE 1. The statements of [2].

Additional result	Source	See
Two-walk Cauchy–Schwarz proof; $N \leq \sqrt{d_H(x)d_H(y)}\, q^{(t-1)/2}$	AB	Cor. 2.2
Path-wise and weighted inequalities, degree-weighted LYM inequality	A	Prop. 2.1
Exact defect (stability) identity	B	Prop. 2.4
Chebyshev proof; link with Carne–Varopoulos; corrected random-walk bound	AB	§3
Refinement $N \leq 2(\rho(H)/2)^t$	B	Prop. 3.1
Explicit equality classification, odd $\Delta \geq 5$	B	Thm. 5.1
Girth bound; cubic extremal graphs of large girth; cover construction	AB / B / B	§6
Link proof of Prop. 4 for all closed surfaces; κ-connected version	AB / B	§7
Planar graphs: exact value 2Δ at distance 3	B	Cor. 8.2
Planar graphs: exact value $\Delta(\Delta+1)$ at distance 4	merge	Prop. 8.6
Planar graphs: $N \leq 2\sqrt{\Delta}\,(\Delta + \sqrt{\Delta} - 1)^{(t-1)/2}$; lower bounds	A	Thm. 8.4, Prop. 8.5

TABLE 2. Results beyond [2] and [8] recorded here.

The walk from x that repeatedly moves along a uniformly random edge of H to the next layer always reaches y after t steps and traverses P with probability $Q_+(P) = 1/F_P$. Thus Q_+ is a probability measure on $\mathcal{P}$; so is $Q_-(P) = 1/G_P$, by symmetry.

Proposition 2.1. (a) $\displaystyle\sum_{P\in\mathcal{P}} (F_P G_P)^{-1/2} \leq 1.$

(b) $\displaystyle N \leq \max_{P\in\mathcal{P}} \frac{2F_P G_P}{F_P + G_P}.$

(c) *Orient the edges of H from x to y. For any weights $c : E(H) \to (0,\infty)$ put*

$$I_c(v) = \sum_{e \text{ into } v} c_e, \qquad O_c(v) = \sum_{e \text{ out of } v} c_e^{-1}.$$

Then

$$\sum_{P \in \mathcal{P}} \Big[O_c(x)\, I_c(y) \prod_{i=1}^{t-1} I_c(v_i) O_c(v_i) \Big]^{-1/2} \le 1.$$

Proof. (a) By Cauchy–Schwarz, $\sum_P \sqrt{Q_+(P)Q_-(P)} \le (\sum_P Q_+(P))^{1/2} (\sum_P Q_-(P))^{1/2} = 1$. (b) $\mu = \frac{1}{2}(Q_+ + Q_-)$ is a probability measure on N points, so $\min_P \mu(P) \le 1/N$. (c) Let the forward walk leave v along e with probability $c_e^{-1}/O_c(v)$ and the backward walk enter v along e with probability $c_e/I_c(v)$. The resulting path measures satisfy $Q_+^c(P) Q_-^c(P) = \prod_{i=1}^t \big(O_c(v_{i-1}) I_c(v_i)\big)^{-1}$, since the c_e cancel; apply Cauchy–Schwarz as in (a). □

Inequality (b) is the Lubell-type bound of Yu and Zhu [8]. Inequality (a) is weaker path by path (harmonic ≤ geometric mean) but it is a statement about a sum, and it is the form that generalises to (c), which drives Section 8.

Corollary 2.2 (Theorem 1 of [2], slightly sharpened)**.**

$$N \le \sqrt{d_H(x)\, d_H(y)}\; q^{(t-1)/2} \le \Delta\, q^{(t-1)/2}.$$

Proof. $F_P G_P = d_H(x) d_H(y) \prod_{i=1}^{t-1} d^-(v_i) d^+(v_i) \le d_H(x) d_H(y) q^{t-1}$ for every P; now use (a). □

Corollary 2.3 (a degree-weighted LYM inequality)**.** *For every multigraph G (no degree bound) and all x, y at distance t,*

$$\sum_{P \in \mathcal{P}} \prod_{i=1}^{t-1} \frac{2}{\deg_G(v_i)} \;\le\; \sqrt{\deg_G(x) \deg_G(y)}.$$

Proof. $d^-(v) d^+(v) \le (\deg_G(v)/2)^2$ and $d_H(x) \le \deg_G(x)$, $d_H(y) \le \deg_G(y)$; use (a). □

Proposition 2.4 (exact defect identity)**.** *Let U be the uniform measure on $\mathcal{P}$. Then*

$$\log \frac{\sqrt{d_H(x) d_H(y)}\, q^{(t-1)/2}}{N} = \frac{1}{2}\, \mathbb{E}_U \sum_{i=1}^{t-1} \log \frac{q}{d^-(v_i) d^+(v_i)} + \frac{1}{2}\, \mathrm{KL}(U \,\|\, Q_+) + \frac{1}{2}\, \mathrm{KL}(U \,\|\, Q_-),$$

and all three terms on the right are non-negative.

Proof. $\mathrm{KL}(U\|Q_+) = \mathbb{E}_U \log F_P - \log N$ and $\mathrm{KL}(U\|Q_-) = \mathbb{E}_U \log G_P - \log N$; add and rearrange. □

This is the entropy proof of [2] with nothing thrown away. It yields stability statements along a typical geodesic: if $\Delta \ge 3$ and $N \ge e^{-\varepsilon} \sqrt{d_H(x) d_H(y)}\, q^{(t-1)/2}$ then $\mathbb{E}_U \#\{i : d^-(v_i) d^+(v_i) < q\} \le 2\varepsilon / \log \frac{q}{q-1}$ and $\mathrm{KL}(U\|Q_+) + \mathrm{KL}(U\|Q_-) \le 2\varepsilon$.

Remark 2.5 (the entropy proof in [2])**.** The argument is sound. The conditioning lists in displays (1)–(2) are garbled, and in (4) the summand $H(E_{i-1} \mid X_i)$ should read $H(E_{i+1} \mid X_i)$, as used in (5). The claim that the uniform random geodesic is a Markov chain is true (given a prefix ending at v, all continuations from v are equally likely), but it is not needed: $H(E_i \mid \text{past}) \le H(E_i \mid X_{i-1})$ because conditioning reduces entropy.

3. The spectral proof, Carne–Varopoulos, and the random-walk remark

Proposition 3.1. *Let $\rho(H)$ be the spectral radius of the adjacency matrix A of H. Then*

$$N \le 2\Big(\frac{\rho(H)}{2}\Big)^t \le 2\Big(\frac{\Delta}{2}\Big)^t.$$

Proof. $(A^j)_{xy} = 0$ for $j < t$ and $(A^t)_{xy} = N$. The Chebyshev polynomial T_t has leading coefficient 2^{t-1} and $|T_t| \le 1$ on $[-1,1]$, so with $\rho = \rho(H)$,

$$2^{t-1}N/\rho^t = [T_t(A/\rho)]_{xy} \le \|T_t(A/\rho)\| \le 1. \qquad \square$$

Proposition 3.2. *Let P be a transition matrix reversible with respect to π, and let $t = d(x,y) \ge 1$ be the distance in the graph of P. Then*

$$p^t(x,y) \le 2^{1-t}\sqrt{\frac{\pi(y)}{\pi(x)}}, \qquad \textit{equivalently} \qquad \sqrt{p^t(x,y)\,p^t(y,x)} \le 2^{1-t}.$$

Proof. The matrix $S = \operatorname{diag}(\sqrt{\pi})\,P\,\operatorname{diag}(1/\sqrt{\pi})$ is symmetric with $\|S\| \le 1$, and

$$(S^j)_{xy} = \sqrt{\pi(x)/\pi(y)}\;p^j(x,y),$$

which vanishes for $j < t$. Hence $2^{t-1}(S^t)_{xy} = [T_t(S)]_{xy} \le 1$. The second form follows from $\pi(x)p^t(x,y) = \pi(y)p^t(y,x)$. $\square$

Remark 3.3 (relation to Carne–Varopoulos)**.** The Carne–Varopoulos inequality [3, 7] (see [6, §13.2]) states $p^n(x,y) \le 2\sqrt{\pi(y)/\pi(x)}\,\Pr[S_n \ge d(x,y)]$ for simple random walk S_n on $\mathbb{Z}$, and Proposition 3.2 is exactly its case $n = d(x,y)$, proved by the same Chebyshev argument. So the bound $2(\Delta/2)^t$ in the abstract of [2] is this classical inequality applied to $P = (A_G + \operatorname{diag}(\Delta - \deg))/\Delta$. Conversely, for simple random walk $p^t(x,y) = \deg(x)^{-1}\sum_P \prod_{i=1}^{t-1}\deg(v_i)^{-1}$, so Proposition 3.2 is literally Corollary 2.3: the two-walk Cauchy–Schwarz argument is an elementary proof of this case of Carne–Varopoulos, and Corollary 2.2 is its integer-sensitive refinement for odd Δ.

Remark 3.4 (the random-walk remark of [2] is false as stated)**.** [2] claims that for a random walk on a weighted graph the probability of going from x to y in $t = d(x,y)$ steps is at most 2^{1-t}.

Example 1. Edges xa, ab, ac, by, cy; simple random walk; $d(x,y) = 3$ and $p^3(x,y) = 1 \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3} > \frac{1}{4}$.

Example 2. A path $x = v_0, \dots, v_t = y$ with conductances $R, R^2, \dots, R^t$ has $p^t(x,y) = (R/(1+R))^{t-1} \to 1$ as $R \to \infty$, so no bound depending only on t can hold.

The error is the division of the path count by Δ^t, which is legitimate only when every vertex has the same total weight. The correct statement is Proposition 3.2; it gives the claimed bound whenever $\pi(x) = \pi(y)$, in particular for regular graphs and symmetric transition matrices.

4. The sharp bounds of Yu and Zhu

Lemma 4.1 ([8])**.** *Let (a_i, b_i), $1 \le i \le r$, be pairs of positive integers with $a_i + b_i \le \Delta$, $A = \prod a_i$, $B = \prod b_i$. Then*

$$\frac{2AB}{A+B} \le \begin{cases} q^{r/2}, & r \text{ even}, \\ 2q^{(r+1)/2}/\Delta, & r \text{ odd}. \end{cases}$$

If $\Delta = 2k+1$, equality holds if and only if every pair is $(k, k+1)$ or $(k+1, k)$ and the numbers of pairs of the two kinds are equal (r even), respectively differ by one (r odd).

Proof. The harmonic mean is at most $\sqrt{AB} \le q^{r/2}$, with equality iff $A = B$ and all $a_i b_i = q$. This settles even r and even Δ (then $2q^{(r+1)/2}/\Delta = q^{r/2}$). Let $\Delta = 2k+1$ and r be odd. If some $a_i b_i < q$, then $a_i b_i \le q'$ where $q' = q-2$ for $k \ge 2$ and $q' = 1$ for $k = 1$; in both cases $\Delta^2 q' < 4q^2$, so $\sqrt{AB} \le q^{(r-1)/2}\sqrt{q'} < 2q^{(r+1)/2}/\Delta$. Otherwise all pairs are of the two kinds, say j of them equal $(k+1, k)$; then $AB = q^r$ and $A + B = q^{\min(j,r-j)}\big(k^m + (k+1)^m\big)$ with $m = |r-2j|$ odd, and $k^m + (k+1)^m \ge q^{(m-1)/2}(2k+1)$ with equality iff $m = 1$. □

Theorem 4.2 ([8]; confirms Conjecture 2 and answers Question 3 of [2])**.** *For multigraphs,*

$$\max N = \begin{cases} \Delta\, q^{(t-1)/2}, & t \text{ odd}, \\ 2\, q^{t/2}, & t \text{ even}, \end{cases}$$

and for simple graphs

$$\max N = \begin{cases} 1, & t = 1, \\ \Delta, & t = 2, \\ \Delta(\Delta-1)\, q^{(t-3)/2}, & t \ge 3 \text{ odd}, \\ 2(\Delta-1)\, q^{(t-2)/2}, & t \ge 4 \text{ even}. \end{cases}$$

Proof of the upper bounds. By Proposition 2.1(b) it suffices to bound the harmonic mean of F_P and G_P, which is increasing in both. For multigraphs $F_P \le \Delta A$ and $G_P \le \Delta B$ with $r = t-1$ pairs $(a_i, b_i) = (d^+(v_i), d^-(v_i))$; apply Lemma 4.1. For simple graphs and $t \ge 3$, $d^-(v_1) = 1$, $d^+(v_1) \le \Delta - 1$, and symmetrically at v_{t-1}, so $F_P \le \Delta(\Delta-1)A$, $G_P \le \Delta(\Delta-1)B$ with $r = t-3$ pairs; $t \le 2$ is trivial. (Sharpness: $C_{2t,\Delta}$ of [2] for multigraphs; Remark 4.4 for simple graphs.) □

Remark 4.3 (the simple-graph example in [2])**.** For $\Delta = 2k$, [2] blows up C_{2t}, replacing vertices by k vertices and edges by $K_{k,k}$, and states that antipodal vertices are joined by k^{t-1} geodesics. The correct count is $2k^{t-1}$ (both arcs). Even so the example is not extremal: for $t \ge 3$ its ratio to the true maximum $2k(2k-1)k^{t-3}$ is $k/(2k-1)$. For instance $\Delta = 4$, $t = 3$ gives 8 against the maximum 12. Note that for even Δ the upper bound $\Delta(\Delta-1)q^{(t-3)/2}$ stated in [2] is exactly the truth for all $t \ge 3$.

Remark 4.4 (explicit extremal simple graphs)**.** For $t = 3$: $K_{\Delta,\Delta}$ minus a perfect matching, with x joined to one side and y to the other. For even Δ and $t \ge 4$: layers of sizes $1, \Delta, 2(\Delta-1), \dots, 2(\Delta-1), \Delta, 1$, with $(\Delta-1, \Delta/2)$-biregular bipartite graphs at the two ends and $\Delta/2$-regular bipartite graphs in between. For odd $\Delta = 2k+1$ and odd $t \ge 5$: sizes $1, \Delta, 2\Delta, \dots, 2\Delta, \Delta, 1$ with (d^-, d^+) alternating between $(k, k+1)$ and $(k+1, k)$ along the layers. For odd Δ and even $t \ge 4$: two parallel channels of $2k$ vertices per layer, carrying the two opposite alternations. All of these were checked by exact count for $3 \le \Delta \le 8$, $3 \le t \le 8$.

5. Equality for odd $\Delta \ge 5$ (multigraphs)

[8] describes the extremal multigraphs completely for even Δ; for odd Δ it records local type conditions and a balance condition along every geodesic. The latter can be made explicit.

Theorem 5.1. *Let $\Delta = 2k+1$ with $k \ge 2$ and $t \ge 2$. Then N equals the maximum in Theorem 4.2 if and only if H has the following form. Each internal layer V_i $(1 \le i \le t-1)$ consists of one vertex A_i with $(d^-, d^+) = (k, k+1)$ and one vertex B_i with $(d^-, d^+) = (k+1, k)$; x is joined to (A_1, B_1) by $(k, k+1)$ edges and (A_{t-1}, B_{t-1}) to y by $(k+1, k)$ edges; the*

multiplicities between (A_i, B_i) *and* (A_{i+1}, B_{i+1}) *are*

$$M_i = \begin{pmatrix} \rho_i & k+1-\rho_i \\ k-\rho_i & \rho_i \end{pmatrix}, \qquad 0 \le \rho_i \le k, \quad 1 \le i \le t-2,$$

and, writing $J = \{i : \rho_i > 0\}$,

- *if* t *is odd: every element of* J *is even;*
- *if* t *is even: there are no* $i < j$ *in* J *with* i *odd and* j *even.*

Proof. By the proof of Theorem 4.2, equality holds iff $\mu(P) = \frac{1}{2}(Q_+(P) + Q_-(P))$ equals the reciprocal of the bound for *every* geodesic, i.e. iff $d_H(x) = d_H(y) = \Delta$ and every geodesic attains equality in Lemma 4.1 with $r = t-1$. As every vertex of H lies on a geodesic, all internal vertices have type A $(k, k+1)$ or B $(k+1, k)$. If V_1 has p vertices of type A and p' of type B, counting the edges at x gives $kp + (k+1)p' = 2k+1$, whose only solution for $k \ge 2$ is $p = p' = 1$. The edges leaving V_1 number $(k+1) + k = 2k+1$, so the same count applies to V_2, and so on; row and column sums force the matrices M_i.

It remains to express "every geodesic has $\#A - \#B = 0$ (t odd), $= \pm 1$ (t even)". A geodesic is a word in $\{A, B\}^r$; at step i the transition $A \to B$ is always available, $A \to A$ and $B \to B$ are available iff $\rho_i > 0$, and $B \to A$ iff $\rho_i < k$. Cut $\{1, \dots, r\}$ after each $i \in J$. Inside a block the types alternate, so a block of even length contributes 0 to $\#A - \#B$ and a block of odd length contributes ± 1. If there are m odd blocks, every geodesic has $|\#A - \#B| \le m$, and the value m is attained: start each odd block with A and each even block with B; then every block ends in A, and both continuations $A \to A$ and $A \to B$ exist at a cut. Since $m \equiv r \pmod 2$, equality holds iff $m = 0$ (r even) or $m = 1$ (r odd, in which case every geodesic has imbalance ± 1). With cut points $0 = j_0 < j_1 < \cdots < j_s < j_{s+1} = r$, a block is odd iff its two cut points differ in parity, which gives the two conditions. □

For $\Delta = 3$ the layer-size equation has further solutions and the extremal family is much richer (see Proposition 6.2). The criterion was confirmed against all 122,781 exact matrix products with $k \in \{2, 3, 4\}$ and at most eight internal layers.

6. Large girth

Proposition 6.1. *Suppose* G *has girth greater than* g*, and let* $s = \lfloor g/2 \rfloor$*. If* $t \ge 2s+1$*, then*

$$N \le \Delta\,(\Delta-1)^s\, q^{(t-1-2s)/2}.$$

Proof. A vertex v with $d(x, v) \le s$ has $d^-(v) = 1$, since two geodesics from x to v would contain a cycle of length at most $2s \le g$. So $d^-(v_i)d^+(v_i) \le \Delta - 1$ for $i \le s$ and for $i \ge t - s$; use Proposition 2.1(a). □

The saving over Theorem 1 is the factor $((\Delta-1)/q)^s$. This is exponential in g for $\Delta \ge 4$, but equal to 1 for $\Delta = 3$; so the claim in [2] of an exponent $t - \Theta(g)$ fails for cubic graphs, and this is not an artefact of the proof:

Proposition 6.2. *For every* g *and every sufficiently large* s *there is a cubic graph of girth at least* g *with vertices* x, y *at distance* $t = 2s+1$ *and* $N = 3 \cdot 2^s = \Delta\, q^{(t-1)/2}$.

Proof. Take two rooted trees of depth s with roots x, y, in which the root has three children and every other non-leaf has two. Each has $L = 3 \cdot 2^{s-1}$ leaves. Join the two leaf sets by a simple 2-regular bipartite graph R. The result is cubic, $d(x, y) = 2s+1$, and geodesics correspond to edges of R, so $N = 2L$.

To raise the girth, let $h < g$ be the current girth and $ab \in E(R)$ an edge on a shortest cycle (every cycle uses an edge of R). Balls of radius $g+1$ in a cubic graph have bounded size, so

for large s there is $cd \in E(R)$ (a, c on the x-side) with all four distances between $\{a, b\}$ and $\{c, d\}$ exceeding g. Replace ab, cd by ad, cb. A cycle through exactly one new edge has length $> g$. A cycle through both consists of the two new edges and either paths $d \rightsquigarrow c$, $b \rightsquigarrow a$ in the old graph minus ab, cd, each of length $\geq h - 1$ (total $\geq 2h$), or paths $d \rightsquigarrow b$, $c \rightsquigarrow a$, each of length $> g$. So no cycle of length $\leq h$ is created and at least one is destroyed; iterate. □

Remark 6.3 (the lower-bound construction in [2])**.** [2] suggests replacing each edge of C_{2t} by a biregular bipartite graph of large girth. This does not make the girth large: two consecutive interfaces can create a 4-cycle $u\, v\, w\, v'$ with $u \in V_i$, $v, v' \in V_{i+1}$, $w \in V_{i+2}$. A construction that works for $\Delta = 2k$: take a $2k$-regular graph F of girth $> g$ with an Eulerian orientation, and join (u, i) to $(v, i+1)$ for every arc $u \to v$. This layered graph covers F, so its girth exceeds g; from $(u_0, 0)$ there are k^t monotone paths to layer t, all geodesic, so some endpoint receives at least $k^t/|V(F)|$ of them. With $|V(F)|$ exponential in g [5] this gives $k^{t-O(g)}$ for $k \geq 2$.

7. Triangulated surfaces

Lemma 7.1. *Let G be simple, v an internal vertex of H with $d(x, v) = i$, and suppose the graph induced on the neighbours of v is κ-connected. Then v has at least κ neighbours w with $d(x, w) = i$; no edge vw of this kind lies on a geodesic, so $d^-(v) + d^+(v) \leq \deg(v) - \kappa$.*

Proof. Split the neighbours of v into N_-, N_0, N_+ according to $d(x, \cdot) = i - 1, i, i + 1$. Both N_- and N_+ are non-empty (the neighbours of v on a geodesic), and there is no edge between them. Hence N_0 separates the neighbourhood graph and $|N_0| \geq \kappa$. Edges of H join consecutive layers. □

In a simplicial triangulation of a closed surface the link of v is a cycle through all its neighbours, so $\kappa = 2$ (concretely: on each of the two arcs of the link between the predecessor and the successor of v the distance to x passes through i).

Theorem 7.2. *If the neighbourhood of every vertex of the simple graph G induces a κ-connected graph, then for $t \geq 3$, with $Q_\kappa = \lfloor \frac{\Delta-\kappa}{2} \rfloor \lceil \frac{\Delta-\kappa}{2} \rceil$,*

$$N \leq \sqrt{d_H(x) d_H(y)}\, (\Delta - \kappa - 1)\, Q_\kappa^{(t-3)/2}.$$

In particular, for a triangulation of any closed surface (orientable or not), $N \leq \Delta(\Delta - 3)\, Q_2^{(t-3)/2}$.

Proof. Apply Proposition 2.1(a): $d^-(v_1) = 1$ and $d^+(v_1) \leq \Delta - \kappa - 1$, symmetrically at v_{t-1}, and $d^- d^+ \leq Q_\kappa$ in between. □

Proposition 4 of [2] is the weaker bound $\Delta\, Q_2^{(t-1)/2}$ for the sphere.

Remark 7.3 (the proof in [2])**.** It asserts that the graph *induced* on the neighbours of x is a cycle. Only the link is a cycle; the induced graph has chords whenever a separating triangle passes through x (this happened at 28% of the vertices of random triangulations we generated). It then treats only neighbours of x and extends "by contracting N towards x", but contraction changes the degrees, so the induction is not justified as written.

Remark 7.4. For triangulated d-manifolds the links are $(d-1)$-spheres, whose graphs should be d-connected by Barnette's theorem [1]; this would give $\kappa = d$. We have not checked the details.

8. Planar graphs

In this section G is a *simple* planar graph (for planar multigraphs nothing is gained: the extremal multigraph $C_{2t,\Delta}$ of [2] is planar). We write $d^-(v), d^+(v)$ for the numbers of parents and children of v in H.

Lemma 8.1 (interface bound)**.** *For $1 \le i \le t-2$, the number of edges of H between V_i and V_{i+1} is at most $|V_i| + |V_{i+1}|$.*

Proof. Contract $V_0 \cup \cdots \cup V_{i-1}$ to a vertex X and $V_{i+2} \cup \cdots \cup V_t$ to a vertex Y (both sets induce connected subgraphs of H) and delete parallel edges. The result is a simple planar bipartite graph on $n = |V_i| + |V_{i+1}| + 2$ vertices containing all e interface edges, an edge from X to every vertex of V_i and an edge from every vertex of V_{i+1} to Y. Hence $e + |V_i| + |V_{i+1}| \le 2n - 4$. □

Corollary 8.2 (distance three)**.** *If $t = 3$ then $N \le \deg(x) + \deg(y) \le 2\Delta$. This is attained for every $\Delta \ge 5$ by a triangulation of the sphere (the Δ-antiprism with both ends capped; for $\Delta = 5$ the icosahedron), and for every $\Delta \ge 3$ by a planar graph (delete the two rim cycles).*

Proof. For $t = 3$ geodesics correspond to edges between V_1 and V_2, and $|V_1| \le \deg(x)$, $|V_2| \le \deg(y)$. □

For comparison, at $t = 3$ Proposition 4 of [2] gives a bound quadratic in Δ. Distance four is treated in Proposition 8.6 below.

Lemma 8.3 (at most two shared children)**.** *Every vertex of H has at most two children w with $d^-(w) \ge 2$, and at most two parents u with $d^+(u) \ge 2$.*

Proof. Suppose $u \in V_i$ has children w_1, w_2, w_3 with further parents $u_1, u_2, u_3 \in V_i \setminus \{u\}$ (not necessarily distinct). Then $1 \le i \le t-2$. The sets $\{u\}$, $V_0 \cup \cdots \cup V_{i-1} \cup \{u_1, u_2, u_3\}$ and $V_{i+2} \cup \cdots \cup V_t$ are disjoint, connected, and each is adjacent to each of w_1, w_2, w_3. This is a $K_{3,3}$ minor. The second statement follows by exchanging x and y. □

Theorem 8.4. *Let G be a simple planar graph of maximum degree Δ, and let $D \ge 3$ be an integer such that $d^-(v) + d^+(v) \le D$ for all internal vertices of H ($D = \Delta$ always works; $D = \Delta - 2$ works for triangulations of the sphere, by Lemma 7.1). For $r \ge 1$ let*

$$\Lambda(r, D) = \max\Big\{ r^2, \quad 2r + D - 3, \quad \Big(2 + \tfrac{\lfloor D/2 \rfloor - 2}{r}\Big)\Big(2 + \tfrac{\lceil D/2 \rceil - 2}{r}\Big) \Big\},$$

the last term being omitted if $D = 3$. Then for every $r \ge 1$

$$N \le \max\{1, \Delta/r\}\ \Lambda(r, D)^{(t-1)/2}.$$

In particular, taking $r = \sqrt{D}/2 + 1$, for $D \in \{\Delta - 2, \Delta\}$

$$N \le 2\sqrt{\Delta}\,\big(D + \sqrt{D} - 1\big)^{(t-1)/2}.$$

Proof. Call an edge $u \to w$ of H *shared* if $d^+(u) \ge 2$ and $d^-(w) \ge 2$; a *private-child* edge if $d^+(u) \ge 2$, $d^-(w) = 1$; a *private-parent* edge if $d^+(u) = 1$, $d^-(w) \ge 2$; and *isolated* if $d^+(u) = d^-(w) = 1$. Give these weights $c = 1,\ r,\ 1/r,\ 1$ respectively and apply Proposition 2.1(c). Let v be internal with $(a, b) = (d^-(v), d^+(v))$.

If $a = 1$, the edge into v is a private-child or an isolated edge, so $I_c(v) \le r$. If $a \ge 2$, the edges into v are shared or private-parent edges, and at most two are shared by Lemma 8.3, so $I_c(v) \le 2 + (a-2)/r$. Symmetrically $O_c(v) \le r$ if $b = 1$ and $O_c(v) \le 2 + (b-2)/r$ if $b \ge 2$. Therefore $I_c(v)O_c(v)$ is at most r^2 if $a = b = 1$; at most $2r + D - 3$ if exactly one of a, b equals 1; and at most $(2 + \frac{a-2}{r})(2 + \frac{b-2}{r})$ if $a, b \ge 2$, which is largest when $a + b = D$ and $|a - b| \le 1$. So $I_c(v)O_c(v) \le \Lambda(r, D)$.

All children of x have $d^- = 1$, so the edges out of x are private-child edges (or one isolated edge) and $O_c(x) \le \max\{1, \Delta/r\}$; likewise $I_c(y) \le \max\{1, \Delta/r\}$.

For $r = \sqrt{D}/2 + 1$ the three terms of Λ are at most $D + \sqrt{D} - 1$: the second equals it, the third is at most $(2 + \frac{D-4}{2r})^2 = D$, and $r^2 \le D + \sqrt{D} - 1$ for $D \ge 3$. Finally $\Delta/r \le 2\sqrt{\Delta}$ since $D \ge \Delta - 2$. □

Only the absence of a $K_{3,3}$ minor was used. On a surface of genus g the same proof works with "two" replaced by $4g + 2$ in Lemma 8.3, giving growth rate $\sqrt{\Delta} + O(g)$; we have not worked out the constants.

Proposition 8.5 (lower bounds)**.** (a) *(Double tree.) For even t there is a simple planar graph of maximum degree Δ with $N = \Delta(\Delta-1)^{t/2-1}$.*

(b) *(Onion.) Let $D \ge 5$, $k_0 \ge 3$ and $h \ge 1$ be integers, and let $t = 2h$. There is a simple planar graph of maximum degree at most $\max\{D, k_0\}$, and a triangulation of the sphere of maximum degree at most $\max\{D+2, k_0\}$, with*

$$N = (1,1,0)\, M^{h-1}\, (k_0, 0, k_0)^{\mathsf{T}}, \qquad M = \begin{pmatrix} D-3 & D-4 & 0 \\ 2 & 2 & 2 \\ D-2 & D-3 & 2 \end{pmatrix}.$$

Consequently $N = \Theta(\lambda_D^{t/2})$, where $\lambda_D = D + 1 - \frac{6}{D+1} + O(D^{-2})$ is the largest root of

$$\lambda^3 - (D+1)\lambda^2 + 6\lambda - 2 = 0.$$

Proof. (a) Two rooted trees in which the root has Δ children and other non-leaves $\Delta - 1$, glued along their leaves in the same cyclic order.

(b) Let V_1 be k_0 children of x in cyclic order. For $1 \le i < h$ give each $u \in V_i$ exactly $D - d^-(u)$ children, forming an interval of the cyclic order of V_{i+1}, the intervals of consecutive vertices overlapping in exactly one vertex. Thus every vertex has $d^- \in \{1, 2\}$ and at least three children. Reflect the picture in V_h to obtain $V_{h+1}, \dots, V_{2h} = \{y\}$; vertices of V_h have degree at most 4. Drawing the layers on concentric circles shows planarity, and adding the cycle through each layer produces a triangulation. For $h = 1$ no expansion occurs: the planar graph is K_{2,k_0}, and the sphere is its bipyramid completion. Their maximum degrees are $\max\{2, k_0\}$ and $\max\{4, k_0\}$ respectively; hence the statement uses degree bounds. For $h \ge 2$ an expanded rim vertex attains degree D, or $D + 2$ after adding the layer cycles.

Let $a(v)$ be the number of geodesic segments from x to v. For a layer put

$$S_A = \sum_{d^-(v)=1} a(v)^2, \qquad S_B = \sum_{d^-(v)=2} a(v)^2, \qquad C = \sum a(v)a(v'),$$

the last sum over cyclically consecutive pairs. A private child of u has $a = a(u)$ and the child shared by consecutive u, u' has $a = a(u) + a(u')$; a direct count gives $(S_A, S_B, C)^{\mathsf{T}} \mapsto M(S_A, S_B, C)^{\mathsf{T}}$ from one layer to the next, and $(S_A, S_B, C) = (k_0, 0, k_0)$ on V_1. By symmetry $N = \sum_{v \in V_h} a(v)^2 = S_A + S_B$ on V_h. M is primitive with the stated characteristic polynomial. □

Proposition 8.6 (distance four)**.** *If $t = 4$ then $N \le \Delta(\Delta+1)$, and $N \le \Delta(\Delta-1)$ for triangulations of the sphere. Both bounds are attained, for every $\Delta \ge 4$ and for every $\Delta \ge 6$ respectively.*

Proof. Let $D \in \{\Delta, \Delta - 2\}$ bound $d^- + d^+$ at internal vertices, as in Theorem 8.4. If $D < 2$ there is no geodesic of length four. If $D = 2$, every internal geodesic vertex has exactly one parent and one child, so $N \le \Delta$, which is stronger than the claimed bound. We may therefore

suppose $D \geq 3$. For $v \in V_2$ write $(d^-(v), d^+(v)) = (1+\alpha_v, 1+\beta_v)$, and let $e_1 = |V_2| + \sum \alpha_v$ and $e_2 = |V_2| + \sum \beta_v$ be the numbers of edges of the two interfaces. Then

$$N = \sum_{v \in V_2} (1+\alpha_v)(1+\beta_v) = e_1 + \sum \beta_v + \sum \alpha_v \beta_v = e_2 + \sum \alpha_v + \sum \alpha_v \beta_v .$$

By Lemma 8.1, $\sum \alpha_v \leq |V_1| \leq \Delta$ and $\sum \beta_v \leq \Delta$. Let p be the number of $u \in V_1$ with $d^+(u) = 1$; the other vertices of V_1 have $d^+ \leq D-1$, so $e_1 \leq \Delta(D-1) - p(D-2)$. By Lemma 8.3, $\alpha_v \leq 1 + p_v$, where p_v is the number of parents of v with $d^+ = 1$, and $\sum_v p_v = p$. Define p', p'_v symmetrically on V_3, so that $e_2 \leq \Delta(D-1) - p'(D-2)$ and $\beta_v \leq 1 + p'_v$. If $\beta_v \geq 1$ then $\alpha_v \leq D-3$, hence

$$\sum \alpha_v \beta_v \leq \sum_{\beta_v \geq 1} \alpha_v (1 + p'_v) \leq \Delta + (D-3) p' ,$$

and likewise with p in place of p'; so $\sum \alpha_v \beta_v \leq \Delta + (D-3)(p+p')/2$. Averaging the two expressions for N,

$$N \leq \Delta(D-1) - \tfrac{(p+p')(D-2)}{2} + \Delta + \Delta + \tfrac{(p+p')(D-3)}{2} = \Delta(D+1) - \tfrac{p+p'}{2} .$$

Equality holds for the onion of Proposition 8.5(b) with $h = 2$ and $k_0 = \Delta$ (for $h = 2$ that construction only needs $D \geq 4$), where $N = k_0(D+1)$. □

D	general $\sqrt{q(D)}$	lower $\sqrt{\lambda_D}$	upper $\min_r \sqrt{\Lambda(r,D)}$	$\sqrt{D+\sqrt{D}-1}$
3	1.414	1.414	1.414	1.932
4	2.000	2.000	2.000	2.236
5	2.449	2.202	2.288	2.497
6	3.000	2.463	2.562	2.729
7	3.464	2.684	2.785	2.940
8	4.000	2.882	3.000	3.135
10	5.000	3.232	3.372	3.487
12	6.000	3.540	3.702	3.803
16	8.000	4.080	4.275	4.359
24	12.000	4.976	5.217	5.282
50	25.000	7.133	7.446	7.488
100	50.000	10.047	10.412	10.440

TABLE 3. Exponential growth rates $\lim N^{1/t}$. For planar graphs read $D = \Delta$; for triangulations of the sphere read $D = \Delta - 2$ (rows $D \geq 4$), the first column then being the rate of Proposition 4 of [2]. For $D = 3, 4$ the lower bounds come from the double tree and the square (resp. triangular) lattice.

So for triangulations of the sphere Proposition 4 of [2] has the right exponential rate only for $\Delta = 6$ (the flat case), and for planar graphs the general bound has the right rate only for $\Delta \leq 4$. In both settings the rate is $\sqrt{\Delta}\,(1+o(1))$: between $\sqrt{D+1-o(1)}$ and $\sqrt{D+\sqrt{D}-1}$.

Remark 8.7 (the spectral route)**.** Proposition 3.1 together with the bound $\rho \leq \sqrt{8\Delta - 16} + 3.47$ of Dvořák and Mohar [4] for planar graphs gives $N \leq 2(\sqrt{2\Delta - 4} + 1.74)^t$ at once. This also shows the order $\sqrt{\Delta}$, with a base worse by a factor about $\sqrt{2}$.

Problem 8.8. Determine the exact growth rate for planar graphs with $\Delta \geq 5$ and for triangulations of the sphere with $\Delta \geq 7$, and the exact maximum for $t \geq 5$. The weights in the proof of Theorem 8.4 are not tight on the onion (they would force $a(\cdot)$ to be constant on layers), so neither bound in Table 3 need be the truth. The onion is extremal for $t = 4$; is it extremal for all even t?

9. The filling questions

Conjecture 5. [2] defines irreducible cycles and then states the conjecture for arbitrary cycles. Irreducibility is presumably meant as a hypothesis: for a sum of far-apart cycles the numbers of minimal fillings multiply, and so do the implied constants. For $d = 1$ and $c = x + y$ the conjecture is Theorem 1 with $c(1) = \frac{1}{2}$. In general the trivial bound is $\Delta^{m(c)}$: reveal a minimal filling by repeatedly taking the first $(d-1)$-face of the current boundary and choosing one of the at most Δ d-faces through it. The difficulty in adapting Section 2 is that there is no layering making "forward" and "backward" choices at a face disjoint. We have no progress to report.

Question 6 asks for the number of minimal fillings of a square of side n in $\mathbb{Z}^3$ with i.i.d. Bernoulli(p) weights on the 2-faces. As written the weights do not enter. Three readings:

- *Minimise total weight.* If weight 0 has positive probability, almost surely there are infinitely many unit cubes all of whose faces have weight 0; adding the boundary of such a cube (disjoint from the support) to a minimiser gives another minimiser. So the number of minimisers is almost surely infinite.
- *Minimise area, ignoring the weights.* The flat square is the unique minimiser: projecting a filling to the plane of the square shows that each of the n^2 cells is covered an odd number of times by horizontal faces, and with exactly n^2 faces there is no room for a change of height.
- *Percolation.* Keep each face with probability p and count minimum-area fillings inside the random complex. This is well posed whenever a filling exists and is perhaps what was intended; we have not analysed it.

10. Numerical checks

All checks are exact counts by dynamic programming over the geodesic graph (scripts accompany these notes).

- Theorem 4.2, Proposition 2.1(a),(b): 22,621 vertex pairs in random multigraphs and simple graphs with $\Delta \leq 6$; the constructions of Remark 4.4 for $3 \leq \Delta \leq 8$, $3 \leq t \leq 8$.
- Propositions 3.1 and 2.4: 14,453 pairs. Proposition 3.2: all pairs in 300 random weighted graphs; Examples 1 and 2 of Remark 3.4.
- Theorem 5.1: 122,781 parameter choices.
- Proposition 6.2: explicit graphs with $(s, \text{girth}) = (5, 6), (7, 8), (8, 10)$.
- Lemma 7.1 with $\kappa = 2$: 68,713 pairs in random triangulations of the sphere. Lemma 8.3 and Theorem 8.4: 148,964 pairs in those triangulations and in random planar subgraphs of them.
- Lemma 8.1: 136,301 interfaces. Corollary 8.2: 34,068 pairs, and capped antiprisms. Proposition 8.6: 20,159 pairs, and onions with $h = 2$. Theorem 7.2 for the sphere: 31,400 pairs.
- Proposition 8.5: explicit onions for $D \in \{5, 6, 8, 10\}$ up to $4 \cdot 10^5$ vertices, compared with the matrix formula and with Theorem 8.4.

11. Open ends

(1) The exact planar and spherical growth rates, and the exact planar maximum for $t \geq 5$.
(2) A structural (rather than along-a-typical-geodesic) stability theorem from Proposition 2.4.
(3) The equality cases for $\Delta = 3$, and for simple graphs.
(4) A sharp large-girth bound for $\Delta \geq 4$.
(5) Conjecture 5 of [2], and a well-posed version of Question 6.

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