

Weakly mixing polygonal billiards

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Abstract

We prove that there exists a G_δ dense set of (non-rational) polygons such the billiard flow is weakly mixing with respect to the Liouville measure (on the unit tangent bundle to the billiard). This follows, via a Baire category argument, from showing that for any translation surface the product of the flows in almost every pair of directions is ergodic with respect to Lebesgue measure. This in turn is proven by showing that for every translation surface the flows in almost every pair of directions do not share non-trivial common eigenvalues.

1. Introduction

A basic question in ergodic theory is the dynamical properties of the billiard flow in various (planar) domains (which we will call ‘tables’) [Kat05], [MT02], [Tab05], [CM06], [Gut12]. Frequently, the table is assumed to have piecewise C^1 boundary, and the billiard is assumed to be a massless point traveling without friction on the interior of the table that experiences elastic collision when it hits the boundary of the table. We will be concerned with such a system where the table is a polygon. The main result of this paper is,

THEOREM 1.1. *There exists a weakly mixing billiard flow in a polygon.*

In fact, we answer a conjecture stated by E. Gutkin and A. Katok (see [GK88], §1) that the set of polygonal tables with a weakly mixing billiard flow is a dense G_δ subset of the appropriate space of all polygonal tables with a fixed number of vertices.

Let P denote a polygon. The billiard flow is a flow $F^t : (P \times S^1)/\sim \rightarrow (P \times S^1)/\sim$, where $(p, \theta) \sim (q, \psi)$ if $p = q \in \partial P$ and the angle between θ and the side of P at p is π minus the angle between ψ and the side of P at p . This

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flow is defined for any orbit that does not orbit into the vertices of P . This flow preserves the Lebesgue measure on $P \times S^1$, defined as the product of the Lebesgue measure on P times the Lebesgue measure on S^1 (pushed forward by $\sim$).

A natural dynamical consequence of this result is that there exists a polygon P of area 1 so that for any rectangles $R, R' \subset P$ and for any intervals $I, I' \subset S^1$, for almost every $p, p' \in P$ and $\theta, \psi \in S^1$ we have

$$\frac{|\{0 \leq t \leq T : F^t(p, \theta) \in R \times I \text{ and } F^t(p', \psi) \in R' \times I'\}|}{T} \rightarrow |R||I| \cdot |R'||I'|.$$

S. Kerckhoff, H. Masur and J. Smillie proved that there were ergodic billiard flows in polygons [KMS86]. The significant general results about billiard flows in polygons are C. Boldrighini, M. Keane and F. Marchetti's result that they have at most a countable set of directions containing periodic trajectories [BKM78] and A. Katok's result that the billiard flow has zero entropy [Kat87, Section 3]. By a result of A. Katok [Kat87, Section 3], the zero entropy property implies a subexponential bound on the growth of complexity, in particular for the counting function of generalized diagonals and periodic orbits. Recently, D. Scheglov [Sch13], [Sch20] improved this bound in the case of almost all triangles to 'weakly exponential'. We recall that H. Masur [Mas90] showed that the counting function of generalized diagonals grows quadratically for rational tables, and it has been conjectured that polynomial bounds should hold for general typical polygons ([Kat87], §4). The most significant recent results on the ergodic properties of billiards in polygons, all of which are in different contexts, are the results of A. Avila and V. Delecroix [AD16] and D. Auricino, A. Avila and V. Delecroix [AAD], which prove weak mixing *in almost all directions* in certain *rational polygons* and the proof by A. Málaga and S. Troubetzkoy [MST17] of the weak mixing property in almost all directions for rational billiards in generic *polygons with vertical-horizontal sides*. In these cases the unit tangent bundle splits into invariant surfaces, and the weak mixing property is proved with respect to the two dimensional measure on the generic invariant surface, as opposed to our setting, in which we prove weak mixing with respect to the (three-dimensional) Liouville measure on the unit tangent bundle. Another recent result is the proof by J. Bobok and S. Troubetzkoy [BT23] of *topological weak mixing* for the billiard map (that is, for the first return map to the sides of the polygon, which is a $\mathbb{Z}$ -action, as opposed to the billiard flows we consider) in the generic polygon.

Similar to the proof of existence of ergodic billiards [KMS86], and of the *directional* weak mixing results of [GK88], our result is obtained from a result about every translation surface via a Baire category argument. A translation surface is a pair (X, ω) , where X is a Riemann surface and ω is an Abelian

differential. From ω we obtain an S^1 family of vector fields on X , which in turn give flows $F_{r_\theta\omega}^t$ on X and a Lebesgue measure on X . We denote this normalized Lebesgue measure λ^2 , regardless of the surface, which is preserved by these flows.

THEOREM 1.2. *For almost every $(\theta, \phi) \in S^1 \times S^1$ the flow $F_{r_\theta\omega}^t \times F_{r_\phi\omega}^t$ is $\lambda^2 \times \lambda^2$ ergodic.*

P. Hubert and the first named author [CH17] previously showed that for almost every surface (with respect to any $SL(2, \mathbb{R})$ -invariant measure) the product of the flow in almost every direction is uniquely ergodic. Our methods for establishing Theorem 1.2 are spectral, as we show the following, which implies Theorem 1.2 because the straight line flow on any translation surface is λ^2 ergodic in almost every direction [KMS86] (see, for instance, [KT06, Prop. 4.2]).

THEOREM 1.3. *For every $\alpha \neq 0$ and every translation surface ω , we have $|\{\theta \in S^1 : F_{r_\theta\omega}^t \text{ has eigenvalue } \alpha\}| = 0$.*

A. Avila and the second named author [AF07] showed that for almost every translation surface (in genus at least 2) the flow in almost every direction is weakly mixing, which implies the above result for almost every surface. In fact, by an announcement of D. Aulicino, A. Avila and V. Delecroix [AAD], Theorem 1.3 holds for almost every surface with respect to every $SL(2, \mathbb{R})$ invariant measure. (They show that for any $SL(2, \mathbb{R})$ ergodic measure that is not supported on branched covers of tori the vertical flow on almost every surface is weakly mixing. This gives Theorem 1.2 for these measures. By an argument based on rigidity sequences, one can show that Theorem 1.3 holds for any branched cover of a torus. In fact, one can prove (see, for instance, [FH19]) that for any $\alpha \in \mathbb{R} \setminus \{0\}$ and for the flow in almost every direction on the branched cover of the torus, there exists a rigidity sequence t_j so that $e^{2\pi i \alpha t_j}$ does not converge to 1).

Note that it is well known that single cylinder surfaces give many examples of surfaces (even with full orbit closure) where every direction has a non-constant eigenfunction.

1.1. Organization of the paper. In Section 1.2 below we gather some fundamental open questions in the ergodic theory of billiards in polygons. In Section 2 we recall some basic material about the Teichmüller flow and the renormalization cocycle for translation flows, the Kontsevich–Zorich cocycle over the Teichmüller flow on the Hodge bundle. We conclude the section, in Section 2.1, with an outline of our argument. In Section 3 we derive several consequences of the work of [EM18], [EMM15] and [CE15], including results on the growth of vectors under the action of the Kontsevich–Zorich cocycle and on averages along horocycle arcs of the pushforward of ‘height’ functions

(functions as in [Theorem 3.8](#)) for the moduli space by the geodesic flow. [Section 4](#) contains, in [Section 4.1](#), preparatory results on the growth of curves in the Hodge bundle, which are derived from results of the previous section, and, in [Section 4.2](#), some standard large deviation results, which are included for convenience of the reader. [Section 5](#) contains the core of the argument, that is, the key proposition on the controlled growth of curves under the Kontsevich–Zorich cocycle ([Proposition 5.1](#)). In [Section 6](#), after recalling Veech’s criterion for weak mixing, we prove our main result, [Theorem 1.3](#). Finally, in [Section 7](#) we derive our result on the existence of weakly mixing polygonal billiards.

1.2. Open questions.

Question 1.4. Does every translation surface ω where the flow is not weakly mixing in almost every direction have the property that the line $\mathbb{R}[\Im(r_\theta\omega)]$ spanned by the cohomology class $[\Im(r_\theta\omega)]$ of the imaginary part of its rotation $r_\theta\omega$ intersects an integer translate of the stable subspace of the $SL(2, \mathbb{R})$ subbundle in a set of directions $\theta \in S^1$ of positive measure?

Question 1.5. For every translation surface is the flow in almost every pair of directions uniquely ergodic? Spectrally singular (modulo constants)?

There are also many natural questions about the flow of billiards in irrational polygons.

Question 1.6. Is there a polygon with a mixing billiard flow? Topologically mixing? Minimal? Is the billiard flow in every irrational polygon ergodic? Weak mixing? Mixing?

Note that numerical experiments [[CP99](#)] suggest that every billiard in an irrational triangle is mixing.

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2. Background

As stated earlier, a translation surface is a pair (X, ω) where X is a finite type Riemann surface and ω is an Abelian differential. The space of translation surfaces is stratified by the order and number of the zeros of the Abelian

differential. Let $\mathcal{H}(\alpha)$ be the moduli space of translation surfaces (X, ω) such that ω has k zeros of orders $\alpha := (\alpha_1, \dots, \alpha_k)$, which we call a stratum. The (real) Hodge bundle of the stratum is a bundle over each stratum whose fiber at (X, ω) is $H^1(X; \mathbb{R})$. Given a translation surface (X, ω) , integrating ω provides charts for $X \setminus \Sigma$ where Σ is the set of zeros of ω . From its action on these charts, the group $SL(2, \mathbb{R})$ of real matrices with determinant one acts on each translation surface, preserving the stratum it is in.

The group $SL(2, \mathbb{R})$ also acts on the Hodge bundle since it acts on the base $\mathcal{H}(\alpha)$ of the bundle, and this action can be lifted to the bundle by parallel transport of cohomology classes with respect to a natural flat connection.

The Kontsevich–Zorich cocycle is given by the action of $SL(2, \mathbb{R})$ on the Hodge bundle. It is a cocycle over the action of the group $SL(2, \mathbb{R})$ on the stratum $\mathcal{H}(\alpha)$. See the lecture notes of the second named author and Carlos Matheus [FM14] for a detailed description of the above material.

Let

$$h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \hat{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

denote, respectively, the Teichmüller horocycle flows and the Teichmüller geodesic flow. Also let

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

denote the group of rotations.

Let $I \subset \mathbb{R}$ be any interval. Given a horocycle arc $\{h_s\omega | s \in I\}$ at $\omega \in \mathcal{H}(\alpha)$, a horocycle section at ω is a map $\phi: I \rightarrow H^1(M, \mathbb{R})$ such that $\phi(s) \in H_{h_s\omega}^1(M, \mathbb{R})$. Here $H^1(M, \mathbb{R})$ denotes the Hodge bundle over the moduli space of Abelian differentials and $H_\omega^1(M, \mathbb{R})$ denotes the fiber at $\omega \in \mathcal{H}(\alpha)$.

For every $s \in I$, let $\pi_\omega(s): H_{h_s\omega}^1(M, \mathbb{R}) \rightarrow H_\omega^1(M, \mathbb{R})$ denote the linear map given by the parallel transport along the horocycle arc joining $h_s\omega$ to ω , that is, $\{h_{s-\sigma}\omega | \sigma \in [0, s]\}$. Given a horocycle section $\phi: I \rightarrow H^1(M, \mathbb{R})$, the curve $\phi_\omega: I \rightarrow H_\omega^1(M, \mathbb{R})$ is defined as the composition

$$\phi_\omega(s) = \pi_\omega(s) \circ \phi(s), \quad \text{for all } s \in I.$$

Since the Kontsevich–Zorich cocycle is defined by parallel transport, the following commutation relation holds: for all $v \in H_{h_s\omega}^1(M, \mathbb{R})$,

$$\pi_{g_t\omega}(e^{2t}s) \circ KZ(g_t, h_s\omega)v = KZ(g_t, \omega) \circ \pi_\omega(s)v.$$

It follows that for any horocycle section $\phi: I \rightarrow H^1(M, \mathbb{R})$, at ω we have the identity

$$KZ(g_t, h_s\omega)\phi(s) = \pi_{g_t\omega}^{-1}(e^{2t}s)KZ(g_t, \omega)\phi_\omega(s).$$

In other terms, since $g_th_s\omega = h_{e^{2t}s}g_t\omega$, the section

$$KZ(g_t, h_{e^{-2t}s}\omega)\phi(e^{-2t}s) = \pi_{g_t\omega}^{-1}(s)KZ(g_t, \omega)\phi_\omega(e^{-2t}s)$$

is a horocycle section at $g_t\omega$. Hence, in order to compute the evolution of horocycle sections under the Teichmüller flow, it is enough to compute the evolution of curves $\phi_\omega: I \rightarrow H_\omega^1(M, \mathbb{R})$ under the maps

$$\phi_\omega(s) \rightarrow KZ(g_t, \omega)\phi_\omega(e^{-2t}s).$$

Let $|\cdot|$ denote the Hodge norm. We recall that the Hodge norm of a cohomology class $\gamma \in H^1(X, \mathbb{C})$ is given by

$$|\gamma| = \sqrt{\frac{1}{2} \int_X \gamma \wedge * \bar{\gamma}},$$

where $*$ denotes the Hodge star operator. This norm depends on the Riemann surface (but not the Abelian differential), though we usually suppress this dependence. See the survey [FMZ14] for a detailed description. For every $g \in SL(2, \mathbb{R})$, let $\|KZ(g, \omega)\|$ denote the operator norm with respect to the Hodge norms on $H_\omega^1(M, \mathbb{R})$ and $H_{g\omega}^1(M, \mathbb{R})$. If $\mathcal{F} \subset H^1(M, \mathbb{R})$ is a subbundle of the (real) Hodge bundle, let $\|KZ(g, \omega)\|_{\mathcal{F}}$ denote the operator norm with respect to the Hodge norm restricted to $\mathcal{F}$.

LEMMA 2.1. *Let dist denote the hyperbolic distance on $SL(2, \mathbb{R})/SO(2)$. For any Abelian differential $\omega \in \mathcal{H}(\alpha)$ and for all $g \in SL(2, \mathbb{R})$, we have $\|KZ(g, \omega)\| \leq e^{\text{dist}(g, Id)}$ and moreover $|KZ(g, \omega)v| \geq e^{-\text{dist}(g, Id)}|v|$.*

Proof. For $g = g_t$ a diagonal element of $SL(2, \mathbb{R})$, the upper bound holds for the Hodge norm by the first variation formulas [For02, §2] or [FM14, §3.5]. The lower bound also follows since $KZ(g_t, \omega)$ is invertible with inverse $K(g_{-t}, g_t\omega)$, hence for $t \geq 0$

$$|v| = |KZ(g_{-t}, g_t\omega)KZ(g_t, \omega)v| \leq e^t |KZ(g_t, \omega)v|.$$

For a general $g \in SL(2, \mathbb{R})$, the result follows from the KAK decomposition of $SL(2, \mathbb{R})$, since the action on the Hodge bundle of the group $K = SO(2) \subset SL(2, \mathbb{R})$ of rotations is isometric. $\square$

LEMMA 2.2. *For every $c > 0$ there is a compact set $\mathcal{K}$ and $t_0 > 0$ so that if $\omega \in \mathcal{K}$, then $|\{-1 \leq s \leq 1 : g_t h_s \omega \in \mathcal{K}\}| > 2 - c$ for all $t > t_0$.*

Proof. By [Ath06, §2.3], there exist V , a continuous, proper function on $\mathcal{H}$, and constants c, γ, a, b, C and τ such that, for all $t \geq \tau$,

$$\int_0^{2\pi} V(g_t r_\theta \omega) d\theta \leq c e^{-\gamma t} V(\omega) + b$$

and, moreover, $a^{-1}V(\omega) < V(g\omega) < aV(\omega)$ for all $g \in SL(2, \mathbb{R})$ with $|g| < C$. By choosing N large enough and using that $r_\theta = \hat{h}_{\tan(\theta)} g_{\log(\cos(\theta))} h_{-\tan(\theta)}$, we have the lemma for $\mathcal{K} = V^{-1}[0, N]$. $\square$

Let ν be an $SL(2, \mathbb{R})$ -invariant measure on a stratum of translation surfaces. We say that $\mathcal{F}$ is a ν -almost everywhere invariant subbundle if, for ν -almost every ω and any $g \in SL(2, \mathbb{R})$, we have that g sends $\mathcal{F}_\omega$ to $\mathcal{F}_{g\omega}$, that is, $KZ(g, \omega)\mathcal{F}_\omega = \mathcal{F}_{g\omega}$.

Following [CE15, Def. 1.3], we say that the KZ cocycle has a ν -measurable almost invariant splitting $\mathcal{F}$ if there exists a finite set of proper subbundles $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{F}$ such that $\mathcal{F}_i \cap \mathcal{F}_j = \{0\}$ ν -almost everywhere, for all $1 \leq i, j \leq n$, and for ν -almost all ω and almost all g , the linear map $KZ(g, \omega)$ sends the set $\{\mathcal{F}_{1,\omega}, \dots, \mathcal{F}_{n,\omega}\}$ to the set $\{\mathcal{F}_{1,g\omega}, \dots, \mathcal{F}_{n,g\omega}\}$. Following [CE15, Def. 1.4], we say that the cocycle acts *strongly irreducibly* on $\mathcal{F}$ with respect to the measure ν if it does not admit any measurable almost invariant splitting.

The span of $\Re(\omega), \Im(\omega)$ (respectively, the real and imaginary part of the Abelian differential $\omega \in \mathcal{H}(\alpha)$) defines a smooth invariant, symplectic subbundle of the Hodge bundle, which is then ν -invariant for any $SL(2, \mathbb{R})$ invariant measure ν on $\mathcal{H}(\alpha)$. We call this the $SL(2, \mathbb{R})$ subbundle. Let $\hat{\mathcal{F}}$ denote its symplectic complement, which is also a ν -almost everywhere invariant subbundle.

2.1. Outline of proof. To prove Theorem 1.3, most of our work is to rule out the ‘weak stable space’ (in the terminology of [AF07]). This is different from the approach of [AD16] and our understanding of the approach of [AAD], where they rule it out for ‘structural’ reasons. It also differs from the approach in [AF07], because it is centered more on moduli space. As in the previous approaches we apply the Veech criterion [Vee84] (Lemma 6.1 of this paper), which morally says that it suffices to show that for all α there exists $c > 0$ and a compact set $\mathcal{K}$ so that for arbitrarily large t such that $g_t r_\theta \omega \in \mathcal{K}$ (we actually need two additional appropriate times in $\mathcal{K}$) we have the following lower bound for the distance from the integer lattice $H^1(M_{g_t \omega}, \mathbb{Z})$ of the image of the vector $\alpha \Im(r_\theta \omega) \in H^1(M, \mathbb{R})$ under the Kontsevich–Zorich cocycle:

$$(1) \quad \|KZ(g_t, r_\theta \omega)(\alpha \Im(r_\theta \omega))\|_{\mathbb{Z}} > c.$$

We show that for every ω this one parameter family of classes are transverse to any integer translate of the stable $SL(2, \mathbb{R})$ -bundle. This allows us to apply Lemma 4.6, which makes the assignment

$$\theta \rightarrow \frac{KZ(g_t, r_\theta \omega)(\alpha \Im(r_\theta \omega))}{|KZ(g_t, r_\theta \omega)(\alpha \Im(r_\theta \omega))|},$$

after rescaling the segment by the geodesic flow which exponentially expands it, closer to a constant curve (for typical θ). This lets us apply Proposition 3.1 (a modification of [CE15]) to have that $\|KZ(g_t, r_\theta \omega)\Im(r_\theta \omega)\|$ typically grows in t . The key Proposition 5.1 shows that, under appropriate assumptions, there exists a $0 < \rho < 1$ such that for an appropriately chosen segment of angles J

we have that for most $\theta \in J$ there is an $s > 0$ such that

$$\|KZ(g_{t+s}, r_\theta \omega)(\alpha \mathfrak{S}(r_\theta \omega))\|_{\mathbb{Z}} > \left(\|KZ(g_t, r_\theta \omega)(\alpha \mathfrak{S}(r_\theta \omega))\|_{\mathbb{Z}} \right)^\rho$$

and $g_{t+s} r_\theta \omega \in \mathcal{K}$. A key step in the proof (Lemma 5.4) is a new large deviations estimate for the measure of the set of directions, on any translation surface, where the Kontsevich–Zorich cocycle grows slower than expected. This complements [ASAE⁺21, Thm. 1.5], which proves a similar result for when the growth of the cocycle is larger than expected. Iterating Proposition 5.1, we can avoid the issues of ‘descendants’ (contrary to [AF07], §3 especially (3.6), although this terminology is not used there). Another difference with [AF07]’s approach is that we treat the ‘weak stable’ and stable subbundles at the same time. We also need to treat the unstable- $SL(2, \mathbb{R})$ subbundle and subbundles where the Kontsevich–Zorich cocycle acts isometrically, which we do by straightforward or standard arguments.

Throughout this proof, we treat horocycles instead of circles because they behave better under the geodesic flow, and we relate horocycles to circles in Section 6.2. Section 7 proves Theorem 1.1 via the previously mentioned, and now standard, Baire Category argument. In this section we follow the approach of [Vor97].

3. Making vectors grow

This section proves Propositions 3.1 and 3.2 and then develops, in a straightforward way, some machinery from [EMM15].

Let $\hat{\mathcal{F}}$ denote the complementary subbundle, that is, the symplectic orthogonal, of the $SL(2, \mathbb{R})$ subbundle and let ν be any $SL(2, \mathbb{R})$ -invariant ergodic probability measure.

PROPOSITION 3.1. (*Chaika–Eskin, [CE15]*) *If $\mathcal{F} \subset \hat{\mathcal{F}}$ is an equivariant subbundle, where the Kontsevich–Zorich cocycle acts strongly irreducibly and with a positive exponent, then there exists $\lambda := \lambda_{\mathcal{F}} > 0$ such that, for all $\delta, \epsilon > 0$, and for all L sufficiently large, there exist $\hat{\epsilon} > 0$ and an open set $\hat{U} := \hat{U}(\delta, L)$ so that $\nu(\hat{U}) > 1 - \delta$, and for all $\omega \in \hat{U}$ and for all parallel horocycle sections v at ω , that is, for the all the maps $v : [-1, 1] \rightarrow \mathcal{F}$ such that $v(s) \in \mathcal{F}_{h_s \omega}$ and $\pi_\omega(s)v(s) = v(0)$, we have*

$$\begin{aligned} \text{Leb}(\{-1 \leq s \leq 1 : |KZ(g_L, h_s \omega)u| \geq e^{\lambda L(1-\epsilon)}|u|\}, & \text{ for all } u \in \mathcal{F}_{h_s \omega} \\ \text{with } \angle(u, v(s)) < \hat{\epsilon}\} > 2 - \delta. \end{aligned}$$

Proposition 3.1 is proved below.

For any compact subset $\mathcal{K}$, for any ω and $t > 0$, let

$$v_{\mathcal{K}}(\omega, t) = |\{\tau \in [0, t] : g_\tau(\omega) \in \mathcal{K}\}|.$$

PROPOSITION 3.2. *Let $\mathcal{F} \subset \hat{\mathcal{F}}$ be an equivariant subbundle of $\hat{\mathcal{F}}$. For any compact subset $\mathcal{K}$ there exists $\lambda := \lambda_{\mathcal{F}, \mathcal{K}} \in [0, 1)$ such that*

- $\sup_{v \in \mathcal{F}_\omega} \frac{|KZ(g_t, \omega)v|}{|v|} \leq \exp(t - (1 - \lambda)v_{\mathcal{K}}(\omega, t));$
- $\inf_{v \in \mathcal{F}_\omega} \frac{|KZ(g_t, \omega)v|}{|v|} \geq \exp(-t + (1 - \lambda)v_{\mathcal{K}}(\omega, t)).$

Proof. Both estimates follow from the first variational formulas for the Hodge norm [For02, §2] or [FM14, §3.5]. $\square$

We now prove Proposition 3.1.

3.1. *Proof of Proposition 3.1.* Let $\lambda > 0$ denote the largest element of the Lyapunov spectrum restricted to $\mathcal{F}$.

Let μ be an $SO(2)$ invariant, compactly supported probability measure on $SL(2, \mathbb{R})$, whose support generates a dense subgroup of $SL(2, \mathbb{R})$. Let $\mathcal{M}$ be any $SL(2, \mathbb{R})$ -invariant suborbifold. We recall that A. Eskin and M. Mirzakhani [EM18] and A. Eskin, M. Mirzakhani and A. Mohammadi [EMM15] proved that all orbit closures of the $SL(2, \mathbb{R})$ action on the moduli space are ‘affine’ suborbifolds supporting a unique ‘affine’ probability invariant measure.

For any $L \in \mathbb{N} \setminus \{0\}$ and $\epsilon > 0$, let $E_{\text{good}}(L, \epsilon)$ be the set of all surfaces $\omega \in \mathcal{M}$ so that for every $v \in \mathcal{F}_\omega$ there exists a subset $H(v) \subset SL(2, \mathbb{R})^L$

$$\mu^L(H(v)) > 1 - \epsilon,$$

and for all $(h_1, \dots, h_L) \in H(v)$, we have

$$e^{(\lambda - \epsilon)L}|v| < |KZ(h_L \cdots h_1, \omega)v| < e^{(\lambda + \epsilon)L}|v|.$$

LEMMA 3.3. (Chaika–Eskin [CE15, Lemma 2.11]) *For all $\epsilon > 0$, we have*

$$\lim_{L \rightarrow \infty} \nu(E_{\text{good}}(L, \epsilon)) = 1.$$

Note that by assumption the cocycle acts strongly irreducibly on $\mathcal{F}$ justifying the application of [CE15]. The continuity of the Hodge norm implies that $E_{\text{good}}(L, \epsilon)$ is open.

To prove the proposition, we need to relate random walks to Teichmüller geodesics via standard techniques:

LEMMA 3.4 (Sublinear Tracking). *There exists $\lambda > 0$ (depending only on μ), and for $\mu^{\mathbb{N}}$ -almost all $\bar{g} = (g_1, \dots, g_n, \dots) \in SL(2, \mathbb{R})^{\mathbb{N}}$ there exists $\bar{\theta} = \bar{\theta}(\bar{g}) \in \mathbb{R}$ such that*

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(g_{\lambda n} r_{\bar{\theta}})(g_n \cdots g_1)^{-1}\| = 0.$$

Furthermore, the distribution of $\bar{\theta}$ is uniform, i.e.,

$$(3) \quad \mu^{\mathbb{N}}(\{\bar{g} \in SL(2, \mathbb{R})^{\mathbb{N}} : \bar{\theta}(\bar{g}) \in [\theta_1, \theta_2]\}) = \frac{|\theta_2 - \theta_1|}{2\pi}.$$

Note that (3) follows from the fact that μ is $SO(2)$ invariant. Since, when $\theta \notin \{\pm\frac{\pi}{2}\}$,

$$(4) \quad r_\theta = \begin{pmatrix} 1 & 0 \\ \tan(\theta) & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \sec(\theta) \end{pmatrix} \begin{pmatrix} 1 & -\tan(\theta) \\ 0 & 1 \end{pmatrix}$$

the previous lemma implies:

LEMMA 3.5 (Sublinear Tracking). *There exists $\lambda > 0$ (depending only on μ) such that for $\mu^{\mathbb{N}}$ -almost all $\bar{g} = (g_1, \dots, g_n, \dots) \in SL(2, \mathbb{R})^{\mathbb{N}}$ there exists $\bar{s} := \bar{s}(\bar{g}) \in (-\infty, +\infty)$ with the property that*

$$(5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \|(g_{\lambda n} h_{\bar{s}})(g_n \dots g_1)^{-1}\| = 0.$$

Furthermore, the distribution of $\bar{s}$ is in the measure class of Lebesgue.

Now let $\hat{E}_{\text{good}}(L, \tilde{\epsilon})$ be the set of all surfaces $\omega \in \mathcal{M}$ so that, for every $v \in \mathcal{F}_\omega$, we have

$$\text{Leb} \left(\left\{ s \in [-1, 1] : e^{(\lambda - \tilde{\epsilon})L} < \frac{|KZ(g_L, h_s \omega) v_{\omega'}(s)|}{|v_{\omega'}(s)|} < e^{(\lambda + \tilde{\epsilon})L} \right\} \right) > 2 - \delta.$$

Lemmas 3.5 and 3.3 imply:

$$\text{LEMMA 3.6. } \textit{The following holds: } \lim_{L \rightarrow \infty} \nu(\hat{E}_{\text{good}}(L, \tilde{\epsilon})) = 1.$$

We conclude the proof of Proposition 3.1. Let $\delta > 0$ and $0 < \tilde{\epsilon} < \frac{1}{2}\epsilon$. By Lemma 3.6, for L large enough we have that $\nu(\hat{E}_{\text{good}}(L, \tilde{\epsilon})) > 1 - \delta$ and $e^{\frac{1}{2}\epsilon L} > 8$.

We then choose $\hat{U} := \hat{E}_{\text{good}}(L, \tilde{\epsilon})$, which is an open set by continuity of the Hodge norm.

By construction, for all $\omega \in U$ and for all parallel horocycle sections v at ω , that is for the all the maps $v: [-1, 1] \rightarrow \mathcal{F}$ such that $v(s) \in \mathcal{F}_{h_s \omega}$ and $\pi_\omega(s)v(s) = v(0)$, we have

$$\text{Leb} \left(\left\{ s \in [-1, 1] : e^{(\lambda - \tilde{\epsilon})L} < \frac{|KZ(g_L, h_s \omega) v(s)|}{|v(s)|} < e^{(\lambda + \tilde{\epsilon})L} \right\} \right) > 2 - \delta.$$

Now let $u \in V_{h_s \omega}$ be a vector such that $\angle(u, v(s)) < \hat{\epsilon}$. We can write $u = av(s) + bv^\perp(s)$ with $v^\perp(s) \perp v(s)$ (with respect to the Hodge inner product), and by assumption we have

$$|a| \geq \frac{1 - \hat{\epsilon}}{2} \frac{|u|}{|v(s)|} \quad \text{and} \quad |b| \leq 2\hat{\epsilon} \frac{|u|}{|v^\perp(s)|}.$$

It follows that, if $\hat{\epsilon} < e^{-L} e^{(\lambda - \tilde{\epsilon})L}/8$, by applying Lemma 2.1, we have

$$|KZ(g_L, h_s \omega) u| \geq e^{(\lambda - \tilde{\epsilon})L} \frac{1 - \hat{\epsilon}}{2} |u| - 2\hat{\epsilon} e^L |u| \geq \frac{e^{(\lambda - \tilde{\epsilon})L}}{8} |u| > e^{(\lambda - \epsilon)L} |u|.$$

The argument is completed.

3.2. Developing.

THEOREM 3.7. (*Eskin–Mirzakhani–Mohammadi*) *Let U be an open set, ν be an $SL(2, \mathbb{R})$ invariant and ergodic measure. For any $\epsilon > 0$, there exists a finite set of invariant manifolds, $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ so that for any $\mathcal{C} \subset \text{supp}(\nu) \setminus \bigcup_{i=1}^n \mathcal{Z}_i$, compact, there exists $T_1 > 0$ so that for all $\omega \in \mathcal{C}$ and for all $T \geq T_1$,*

$$\frac{1}{2T} \int_{-1}^1 \int_0^T \chi_U(g_t h_s \omega) dt ds > \nu(U) - \epsilon.$$

This follows from [EMM15, Thm. 2.7] by choosing $\phi \in C_c(\mathcal{H})$, with $\text{supp}(\phi) \subset U$, $0 \leq \phi \leq 1$ with $\|\phi\|_1 > \nu(U) - \frac{\epsilon}{2}$.

THEOREM 3.8. ([EMM15, Prop. 2.13]) *Let $\mathcal{M} \subset \mathcal{H}$ be an affine invariant submanifold. (In this proposition $\mathcal{M} = \emptyset$ is allowed.) Then there exists an $SO(2)$ -invariant function $f_{\mathcal{M}}: \mathcal{H} \rightarrow [1, \infty]$ with the following properties:*

- (1) *$f_{\mathcal{M}}(\omega) = \infty$ if and only if $\omega \in \mathcal{M}$, and $f_{\mathcal{M}}$ is bounded on compact subsets of $\mathcal{H} \setminus \mathcal{M}$. For any $\rho > 0$, the set $\{\omega : f_{\mathcal{M}}(\omega) \leq \rho\}$ is a compact subset of $\mathcal{H} \setminus \mathcal{M}$.*
- (2) *There exists $b > 0$ (depending on $\mathcal{M}$) and for every $0 < c < 1$ there exists $t_0 > 0$ (depending on $\mathcal{M}$ and c) such that for all $\omega \in \mathcal{H} \setminus \mathcal{M}$ and all $t > t_0$,*

$$\frac{1}{2\pi} \int_0^{2\pi} f_{\mathcal{M}}(g_t r_{\theta} \omega) d\theta \leq c f_{\mathcal{M}}(\omega) + b.$$

- (3) *There exists $\sigma > 1$ and $V \subset SL(2, \mathbb{R})$ a neighborhood of the identity so that for all $g \in V$ and all $\omega \in \mathcal{H}$,*

$$\sigma^{-1} f_{\mathcal{M}}(\omega) \leq f_{\mathcal{M}}(g\omega) \leq \sigma f_{\mathcal{M}}(\omega).$$

This implies a similar result for horocycles:

LEMMA 3.9. [ASAE⁺21, Lemma 3.5] *Let $f_{\mathcal{M}}$ be as in Theorem 3.8. Then there exists a constant $b' > 0$ so that for all $0 < a < 1$ there exists $\bar{t}_0 = \bar{t}_0(a)$ such that for all $t > \bar{t}_0$ and for all $\omega \in \mathcal{H} \setminus \mathcal{M}$ we have*

$$\int_{-1}^1 f_{\mathcal{M}}(g_t h_s \omega) ds < a f_{\mathcal{M}}(\omega) + b'.$$

For each $\rho \in \mathbb{R}^+$, let $\mathfrak{C}_{\rho} = \{\omega : f_{\mathcal{M}}(\omega) \leq \rho\}$ and, for all $N \in \mathbb{N}$, let

$$Z_N = \left\{ s \in [-1, 1] : g_{jt} h_s \omega \notin \mathfrak{C}_{\rho} \text{ for all } j \in \{1, 2, \dots, N\} \right\}.$$

PROPOSITION 3.10. [ASAE⁺21, Prop. 3.7] *Let $f_{\mathcal{M}}$ be as in Theorem 3.8, and let $b' > 0$ and $\bar{t}_0 = \bar{t}_0(a)$ be as in Lemma 3.9. There exist $C_1 > 1$ (independent of ω and a) such that for all $a \in (0, 1)$, all $\rho > C_1 b'/a$, all $t \geq \bar{t}_0$*

such that $e^t \in \mathbb{N}$ and all $N \in \mathbb{N}$,

$$\int_{Z_{N-1}} f_{\mathcal{M}}(g_{Nt}h_s\omega)ds < (2a)^N f_{\mathcal{M}}(\omega) + (2a)^{N-1}b'.$$

By choosing $a < \min\{\frac{1}{2}, \frac{1}{2b'}\}$ we obtain:

COROLLARY 3.11. (cf. [Ath06, Theorem 1.1 (2)]) *There exists $\rho \in \mathbb{R}, \zeta < 1$ so that for all S, T large enough if $\omega \in \mathfrak{C}_\rho$*

$$\text{Leb}(\{s \in [-1, 1] : \cup_{t \in [S, S+T]} \{g_t h_s \omega\} \cap \mathfrak{C}_\rho = \emptyset\}) < \zeta^T.$$

PROPOSITION 3.12. [ASAE⁺21, Prop. 3.9] (cf. [KKLM17, Thm. 1.5]) *Let $\mathfrak{C}_\rho$ be as above for a function $f_{\mathcal{M}}$ as in Theorem 3.8. Let us assume that $\omega \notin \mathcal{M}$. For any $\delta, a \in (0, 1)$, there exist $\rho_0 > 1$ and $t_0 > 1$, depending only on a , such that for all $\rho \geq \rho_0$, all $t \geq t_0$ such that $e^t \in \mathbb{N}$, and all $N \in \mathbb{N}$, the set*

$$\left\{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{\mathfrak{C}_\rho}(g_{jt}h_s\omega) < 1 - \delta \right\}$$

can be covered by at most $2^N C_1^N (2a)^{\delta N} e^{2tN} C(\omega)$ intervals of radius e^{-2tN} , where $C(\omega) = \max\{1, f_{\mathcal{M}}(\omega)/\rho\}$.

Note that, if

$$a < \frac{1}{2(2C_1)^{\frac{1}{\delta}}},$$

the measure of the set in the statement of Proposition 3.12 decays exponentially in N .

COROLLARY 3.13. *Let U be an open set whose boundary has measure zero. For all $\epsilon > 0$ there exist numbers $\xi \in (0, 1)$, $t_0 > 0$, $N_0 \in \mathbb{N}$ and a set $V_\epsilon := V_\epsilon(U) \subset \text{supp}(\nu)$, open in $\text{supp}(\nu)$, and $T_2 \in \mathbb{R}^+$ so that for any $T > T_2$:*

- (i) $\nu(V_\epsilon) > 1 - \epsilon$;
- (ii) *for ν almost every ω we have $\liminf_{S \rightarrow \infty} \frac{|\{0 \leq t \leq S : g_t \omega \in V_\epsilon\}|}{S} > 1 - \epsilon$;*
- (iii) *if $\omega \in V_\epsilon$ we have $|\frac{1}{2T} \int_{-1}^1 \int_0^T \chi_U(g_t h_s \omega) dt ds - \nu(U)| < \epsilon$;*
- (iv) *for all $t \geq t_0$, $N \geq N_0$ and $\omega \in V_\epsilon$ we have*

$$\left| \left\{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{V_\epsilon}(g_{jt}h_s\omega) < 1 - \epsilon \right\} \right| < \xi^N.$$

This uses the following standard definition and result:

Definition 3.14. Let $\omega \in \mathcal{H}$ and ν be the unique, $SL(2, \mathbb{R})$ -invariant and ergodic probability measure with $\text{supp}(\nu) = \overline{SL(2, \mathbb{R})\omega}$. We say ω is *Birkhoff*

generic if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \phi(g_t \omega) dt = \int_{\mathcal{H}} \phi d\nu$$

for all $\phi \in C_c(\mathcal{H})$.

LEMMA 3.15. *Let S be a set so that $\nu(\partial S) = 0$. If ω is Birkhoff generic then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi_S(g_t \omega) dt = \nu(S).$$

If U is an open set and ω is Birkhoff generic

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \chi_U(g_t \omega) dt \geq \nu(U).$$

Proof of Corollary 3.13. First, (ii) follows free of charge by the fact that V_ϵ is open and Lemma 3.15. Given an open set U and $\epsilon > 0$, we then construct an open set V_ϵ with properties (i), (iii) and (iv).

For the given U and $\epsilon > 0$, we apply Theorem 3.7 to obtain a finite number of manifolds $\mathcal{Z}_1, \dots, \mathcal{Z}_n$. For each of these manifolds we build a function f_i as in Theorem 3.8 and let $\mathfrak{C}_s^{(i)} = \{\omega : f_i(\omega) \leq s\}$.

Applying Proposition 3.12 to each f_i with $\delta < \frac{\epsilon}{n}$ and $a < \frac{1}{2(2C_1)^{\frac{1}{\delta}}}$ (where C_1 is as in the proposition), there exists ρ_i, t_i as in the proposition, and so that $\nu(\cap \mathfrak{C}_{\rho_i}^{(i)}) > 1 - \epsilon$. Following the sentence after the proposition, for any $\sigma > 1$ there exist constants $\zeta_1, \dots, \zeta_n < 1$ and natural numbers $N_1, \dots, N_n$ so that for each $i \in \{1, \dots, n\}$, for all $t \geq t_i$, for all $\omega \in \mathfrak{C}_{\sigma \rho_i}^{(i)}$ and $N \geq N_i$,

$$\left| \left\{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{\mathfrak{C}_{\rho_i}^{(i)}}(g_{jt} h_s \omega) < 1 - \frac{\epsilon}{n} \right\} \right| < \zeta_i^N.$$

Now, letting $\xi > \max\{\zeta_i\}$, there exist N_0, t_0 so that, if $\omega \in \mathfrak{C}_{\sigma \rho_i}^{(i)}$ for all $i \in \{1, \dots, n\}$, then for all $N \geq N_0$ and $t \geq t_0$ we have

$$\left| \left\{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{\cap \mathfrak{C}_{\rho_i}^{(i)}}(g_{jt} h_s \omega) < 1 - \epsilon \right\} \right| < \xi^N.$$

We define the set V_ϵ to be the interior in $\text{supp}(\nu)$ of the compact set $\mathcal{C} = \text{supp}(\nu) \cap \cap_{i=1}^n \mathfrak{C}_{\sigma \rho_i}^{(i)}$, so that property (iv) holds by the above argument. By applying Theorem 3.7 to the compact set

$$\mathcal{C} := \text{supp}(\nu) \cap \bigcap_{i=1}^n \mathfrak{C}_{\sigma \rho_i}^{(i)}$$

and the given $\epsilon > 0$, we conclude that there exists $T_2 > 0$ such that property (iii) holds for all $T > T_2$. Finally, it follows from the definitions that $\nu(V_\epsilon) \geq \nu(\cap \mathfrak{C}_{\rho_i}^{(i)}) > 1 - \epsilon$, hence property (i) is also proved. $\square$

We will of course be interested in V_ϵ chosen for $U = \hat{U}$, and this is what $\hat{V}_\epsilon$ denotes for the remainder of the paper. We further assume that $\hat{V}_\epsilon$ is contained in a fixed compact set.

4. Some preparatory results

4.1. *Making curves grow.* The main result of the next section is [Lemma 4.8](#), which applies [Proposition 3.1](#) to say that the geodesic flow image of (projectively) Lipschitz horocycle sections with small enough (projective) Lipschitz constants in a subbundle as in [Proposition 3.1](#) typically grow. It also collects some results that say that the geodesic flow image of horocycle sections have their Lipschitz (and usually their projective Lipschitz) constants improve.

Definition 4.1. A horocycle section $\phi: [a, b] \rightarrow H^1(M, \mathbb{R})$ at ω (that is, a map $\phi: [a, b] \rightarrow H^1(M, \mathbb{R})$ such that $\phi(s) \in H_{h_s\omega}(M, \mathbb{R})$ for all $s \in [a, b]$) is K -Lipschitz at ω if the curve $\phi_\omega: [a, b] \rightarrow H_\omega^1(M, \mathbb{R})$, obtained by parallel transport along the horocycle, (that is, such that $\phi_\omega(s) = \pi_\omega(s) \circ \phi(s)$ for all $s \in [a, b]$) is Lipschitz with respect to the Hodge norm on $H_\omega^1(M, \mathbb{R})$ with Lipschitz constant $K > 0$.

LEMMA 4.2. *If $\phi: [a, b] \rightarrow H^1(M, \mathbb{R})$ is a K -Lipschitz horocycle section at ω , then $KZ(g_t)(\phi): [e^{2t}a, e^{2t}b] \rightarrow H^1(M, \mathbb{R})$ given by*

$$KZ(g_t)(\phi)(s) = KZ(g_t, h_{se^{-2t}}\omega)\phi(se^{-2t})$$

is a Ke^{-t} -Lipschitz horocycle section at $g_t\omega$.

Proof. By definition $\phi: [a, b] \rightarrow H^1(M, \mathbb{R})$ is a K -Lipschitz horocycle section at ω if the parallel transport $\phi_\omega: I \rightarrow H_\omega^1(M, \mathbb{R})$ is a K -Lipschitz map with respect to the Hodge norm. We recall that the map ϕ_ω is defined, for all $s \in I$, as a composition $\phi_\omega(s) = \pi_\omega(s) \circ \phi(s)$ with the parallel transport $\pi_\omega(s): H_{h_s\omega}^1(M, \mathbb{R}) \rightarrow H_\omega^1(M, \mathbb{R})$.

By the commutativity of parallel transport and the KZ cocycle, we have

$$\pi_{g_t\omega}(s)KZ(g_t, h_{se^{-2t}}\omega)\phi(se^{-2t}) = KZ(g_t, \omega)\phi_\omega(se^{-2t}),$$

hence by [Lemma 2.1](#) the map defined as $KZ(g_t, \phi)_{g_t\omega}(s) = KZ(g_t, \omega)\phi_\omega(se^{-2t})$ is Ke^{-t} -Lipschitz. $\square$

Definition 4.3. A horocycle section $\phi: [a, b] \rightarrow H^1(M, \mathbb{R})$ at ω is projectively κ -Lipschitz at ω if there exists $K > 0$ such that the curve $\phi_\omega: [a, b] \rightarrow H_\omega^1(M, \mathbb{R})$ is K -Lipschitz at ω and the Hodge norm $\|\phi_\omega(s)\|_\omega$ is bounded below by $K/\kappa > 0$, for all $s \in [a, b]$.

Remark 4.4. Using [Lemma 2.1](#) to bound the change of Hodge norm from above and below under parallel transport along a horocycle, we obtain that there exists $C_{a,b}$, depending only on $b - a$ so that for any section $\phi: [a, b] \rightarrow$

$H^1(M, \mathbb{R})$ projectively κ -Lipschitz at ω , and for any $s \in [a, b]$, the section ϕ is also projectively $C_{a,b}\kappa$ -Lipschitz at $h_s\omega$.

Remark 4.5. For any $K > 0$, every K -Lipschitz horocycle section $\phi : [a, b] \rightarrow H^1(M, \mathbb{R})$ at ω , is projectively κ -Lipschitz at ω if $\max_{s \in [a, b]} \|\phi_\omega(s)\|_\omega > K(b - a)$ with

$$\kappa := K / \left(\max_{s \in [a, b]} \|\phi_\omega(s)\|_\omega - K(b - a) \right).$$

LEMMA 4.6. *If $\phi : [a, b] \rightarrow H^1(M, \mathbb{R})$ is a projectively κ -Lipschitz horocycle section at ω , then $KZ(g_t)(\phi) : [e^{2t}a, e^{2t}b] \rightarrow H^1(M, \mathbb{R})$ given by $KZ(g_t)(\phi)(s) = KZ(g_t, h_{se^{-2t}}\omega)\phi(se^{-2t})$ is a projectively κ -Lipschitz horocycle section at $g_t\omega$. If in addition $\phi : [a, b] \rightarrow \mathcal{F} \subset \hat{\mathcal{F}}$, then for any compact set $\mathcal{K}$ there exists $\lambda_{\mathcal{F}, \mathcal{K}} \in [0, 1)$ such that the following holds: For any ω and for any $t > 0$, let*

$$v_{\mathcal{K}}(\omega, t) := |\{t' \in [0, t] : g_{t'}\omega \in \mathcal{K}\}|.$$

Then for all $t > 0$ the horocycle section $KZ(g_t)(\phi) : [e^{2t}a, e^{2t}b] \rightarrow \hat{\mathcal{F}}$ is projectively κ_t -Lipschitz at $g_t\omega$, where

$$\kappa_t \leq \kappa \exp \left(- (1 - \lambda_{\mathcal{F}, \mathcal{K}}) v_{\mathcal{K}}(\omega, t) \right).$$

Proof. By Lemma 4.2, if the horocycle section ϕ is K -Lipschitz at ω , then $KZ(g_t)(\phi)_{g_t\omega}$ is Ke^{-t} -Lipschitz, and by Lemma 2.1 it follows that

$$\|KZ(g_t)(\phi)_{g_t\omega}(s)\| \geq e^{-t} \|\pi_\omega(\phi)(s)\|$$

for all $s \in [a, b]$. It follows immediately that if $\|\pi_\omega(\phi)(s)\|_\omega \geq K/\kappa$, then

$$\|KZ(g_t)(\phi)_{g_t\omega}(s)\| \geq e^{-t} \|\phi_\omega(s)\| \geq Ke^{-t}/\kappa,$$

hence $KZ(g_t)(\phi)$ is projectively κ -Lipschitz at $g_t\omega$ establishing the first claim.

If $\phi : [a, b] \rightarrow \hat{\mathcal{F}}$, it follows from Proposition 3.2 that there exists $\lambda := \lambda_{\mathcal{F}, \mathcal{K}} \in [0, 1)$ such that

$$(6) \quad \|KZ(g_t)(\phi)_{g_t\omega}(s)\| = \|KZ(g_t, \omega)(\phi_\omega(s))\| \geq e^{-t+(1-\lambda)v_{\mathcal{K}}(\omega, t)} \|\phi_\omega(s)\|.$$

Since ϕ is projectively κ -Lipschitz at ω , there exists $K > 0$ such that ϕ is K -Lipschitz at ω and satisfies the lower bound $\|\phi_\omega(s)\| \geq K/\kappa$, for all $s \in [a, b]$. It follows that

$$(7) \quad \|KZ(g_t)(\phi)_{g_t\omega}(s)\| \geq (K/\kappa)e^{-t+(1-\lambda)v_{\mathcal{K}}(\omega, t)} = Ke^{-t}/(\kappa e^{-(1-\lambda)v_{\mathcal{K}}(\omega, t)}),$$

which implies that the section $KZ(g_t)(\phi)$ is projectively κ_t -Lipschitz at $g_t\omega$ with $\kappa_t \leq \kappa e^{-(1-\lambda)v_{\mathcal{K}}(\omega, t)}$, as stated. $\square$

LEMMA 4.7. *Let $\mathcal{F} \subset \hat{\mathcal{F}}$ be a subbundle where KZ acts strongly irreducibly and with a positive exponent. There exists $\lambda > 0$ such that the following holds: For any $\delta, \epsilon > 0$ and for all sufficiently large $L > 0$, there exist $\kappa_0 := \kappa_0(a, b, \delta, L)$ and an open set $\hat{U} := \hat{U}(\delta, L)$ (as in Proposition 3.1) with $\nu(\hat{U}) >$*

$1 - \delta$, such that for all $\kappa \in (0, \kappa_0)$, for all $t \geq 0$, for all $s_0 \in [a + e^{-2t}, b - e^{-2t}]$ and for all horocycle sections $\phi : [a, b] \rightarrow \mathcal{F} \subset \hat{\mathcal{F}}$ projectively κ -Lipschitz at $h_{s_0}\omega$, whenever $g_t h_{s_0}\omega \in \hat{U}$, we have that

$$(8) \quad \text{Leb}(\{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] :$$

$$|KZ(g_{t+L}, h_s\omega)\phi(s)| < e^{\lambda L(1-\epsilon)} |KZ(g_t, h_s\omega)\phi(s)|\}) < \delta e^{-2t}.$$

Proof. The lemma follows by [Proposition 3.1](#) applied to the horocycle $\{h_s g_t(h_{s_0}\omega) | s \in [-1, 1]\}$ at $g_t h_{s_0}\omega$. Let us consider the image $KZ(g_t)(\phi)$ of the section ϕ for $s \in [s_0 - e^{-2t}, s_0 + e^{-2t}]$. By definition, this section is the restriction of a horocycle section at $g_t\omega$, and after reparametrization it can be regarded as a horocycle section at $g_t h_{s_0}\omega = h_{e^{2t}s_0} g_t\omega$, defined as $KZ(g_t, h_{s_0+e^{-2t}s}\omega)\phi(s_0 + e^{-2t}s)$ for $s \in [-1, 1]$. Since ϕ is projectively κ -Lipschitz at $h_{s_0}\omega$, [Lemma 4.6](#) implies that $KZ(g_t)(\phi)$, hence its reparametrization, are also projectively κ -Lipschitz at $g_t h_{s_0}\omega$. It follows that the ratio K/m between the Lipschitz constant K and the minimum Hodge norm m of the curve

$$KZ(g_t)(\phi)_{g_t h_{s_0}\omega}(s) = \pi_{g_t h_{s_0}\omega}(KZ(g_t)(\phi)(s)), \quad \text{for } s \in [-1, 1],$$

is at most κ , hence, for all $s \in [-1, 1]$ we have that

$$\angle(KZ(g_t)(\phi)_{g_t h_{s_0}\omega}(s), KZ(g_t)(\phi)_{g_t h_{s_0}\omega}\phi(s_0)) < \kappa.$$

Hence there exists $\kappa_0 > 0$ such that, for $\kappa \in (0, \kappa_0)$, the horocycle section

$$KZ(g_t)(\phi)(e^{2t}s_0 + s) = KZ(g_t, h_{s_0+e^{-2t}s}\omega)\phi(s_0 + e^{-2t}s)$$

makes an angle at most $\hat{\epsilon}$ (with respect to the Hodge norm on $H^1_{h_s g_t h_{s_0}\omega}(M, \mathbb{R})$) with a parallel section, so that, since

$$KZ(g_{t+L}, h_{s_0+e^{-2t}s}\omega)\phi(s_0 + e^{-2t}s) = KZ(g_L, h_s g_t h_{s_0}\omega) KZ(g_t)(\phi)(e^{2t}s_0 + s)$$

by [Proposition 3.1](#), we have

$$\begin{aligned} \text{Leb}(\{-1 \leq s \leq 1 : |KZ(g_{t+L}, h_{s_0+e^{-2t}s}\omega)\phi(s_0 + e^{-2t}s)| \\ \geq e^{\lambda L(1-\epsilon)} |KZ(g_t, h_{s_0+e^{-2t}s}\omega)\phi(s_0 + e^{-2t}s)|\}) > 2 - \delta, \end{aligned}$$

which implies the statement by change of variables. $\square$

The main result of this section is the following lemma: Let $\delta > 0$ and let $L > 0$ be sufficiently large.

Let $\hat{U} := \hat{U}(\delta, L)$ be the open set given by [Proposition 3.1](#) and [Lemma 4.7](#), and, for all small $\epsilon > 0$, let $\hat{V}_\epsilon := V_\epsilon(\hat{U}(\delta, L)) \subset \text{supp}(\nu)$ denote the open subset (which depends on $\hat{U}(\delta, L)$) given by [Corollary 3.13](#).

Notation: For all $\epsilon, \delta > 0$, let

$$(9) \quad \begin{aligned} \eta(\epsilon, \delta) &:= \sqrt{3(\epsilon + \delta)} + 2\sqrt{\delta}, \quad \text{and} \\ \mu(\epsilon, \delta) &:= \lambda(1 - \epsilon) - (1 + \lambda(1 - \epsilon))\eta(\epsilon, \delta). \end{aligned}$$

LEMMA 4.8. *Let $\mathcal{F} \subset \hat{\mathcal{F}}$ be a subbundle where KZ acts strongly irreducibly with a positive exponent. For any bounded interval $[a, b] \subset \mathbb{R}$, for all $\delta > 0$, for all $L > 0$ sufficiently large, there exists $\kappa_0 := \kappa_0(a, b, \delta, L)$ such that the following holds: For all $\kappa \in (0, \kappa_0)$, there exists $T_3 > 0$ and $t_0 > 0$ such that, for all $s_0 \in [a + e^{-2t}, b - e^{-2t}]$, whenever*

- $g_t h_{s_0} \omega \in \hat{V}_\epsilon := V_\epsilon(\hat{U}(\delta, L))$,
- $\phi : [a, b] \rightarrow \mathcal{F}$ is a projectively κ -Lipschitz horocycle section at ω ,
- $t > t_0$ and $T > T_3$

we have

$$(10) \quad \text{Leb} \left(\left\{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : |KZ(g_{t+T}, h_s \omega) \phi(s)| \right. \right. \\ \left. \left. < e^{\mu(\epsilon, \delta)T} |KZ(g_t, h_{s_0} \omega) \phi(s)| \right\} \right) < \eta(\epsilon, \delta) e^{-2t}.$$

Proof. Let $\delta > 0$ and $L > 0$ sufficiently large be fixed. By [Corollary 3.13](#) (iii) (and the fact that $\nu(\hat{U}) > 1 - \delta$), since $g_t h_{s_0} \omega \in \hat{V}_\epsilon$, we have that, for all $T > T_2$,

$$\int_0^T \int_{-1}^1 \chi_{\hat{U}}(g_\ell h_s g_t h_{s_0} \omega) ds d\ell \\ = e^{2t} \int_0^T \int_{-e^{-2t}}^{e^{-2t}} \chi_{\hat{U}}(g_\ell g_t h_{s_0+s} \omega) ds d\ell > 2T(1 - \delta - \epsilon).$$

That is,

$$\sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} \int_0^L e^{2t} \int_{-e^{-2t}}^{e^{-2t}} \chi_{\hat{U}}(g_\ell g_{jL} g_t h_{s_0+s} \omega) ds d\ell > 2T - 2T(\epsilon + \delta) - L.$$

From this we have that there exists $a \in [0, L]$ so that

$$\sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} e^{2t} \int_{-e^{-2t}}^{e^{-2t}} \chi_{\hat{U}}(g_{t+a+jL} h_{s_0+s} \omega) ds > \frac{2T - 2T(\epsilon + \delta) - L}{L}.$$

It follows that, except for a set of $s \in [s_0 - e^{-2t}, s_0 + e^{-2t}]$ with measure at most $e^{-2t} \sqrt{2(\epsilon + \delta) + \frac{L}{T}}$, we have

$$\sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} \chi_{\hat{U}}(g_{t+a+jL} h_s \omega) > \left(1 - \sqrt{2(\epsilon + \delta) + \frac{L}{T}} \right) \lfloor \frac{T}{L} \rfloor.$$

For $t \geq t_0$ and for all $s \in [s_0 - e^{-2t}, s_0 + e^{-2t}]$, let

$$J(s) := \# \left\{ j \in \{0, \dots, \lfloor \frac{T}{L} \rfloor - 1\} : g_{t+a+jL} h_s \omega \in \hat{U} \right.$$

$$\left. \text{but } |KZ(g_{t+a+(j+1)L}, h_s \omega) \phi(s)| < e^{\lambda L(1-\epsilon)} |KZ(g_{t+a+jL}, h_s \omega) \phi(s)| \right\}.$$

We claim that

$$(11) \quad \text{Leb}\left(\left\{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : J(s) > 2\sqrt{\delta} \lfloor \frac{T}{L} \rfloor\right\}\right) < 2\sqrt{\delta} e^{-2t}.$$

In fact, for each j , let χ_j denote the characteristic function of the set

$$(12) \quad \{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : g_{t+a+jL} h_s \omega \in \hat{U} \\ \text{but } |KZ(g_{t+a+(j+1)L}, h_s \omega) \phi(s)| < e^{\lambda L(1-\epsilon)} |KZ(g_{t+a+jL}, h_s \omega) \phi(s)|\}.$$

Since the interval $[a, b]$ is bounded, by [Remark 4.4](#) there exists a constant $C_{a,b} > 0$ such that if the section $\phi : [a, b] \rightarrow H^1(M, \mathbb{R})$ is projectively κ -Lipschitz at ω , then ϕ is projectively $C_{a,b}\kappa$ -Lipschitz at $h_s \omega$, for all $s \in [a, b]$. It then follows from [Lemma 4.6](#) that, for all $j \in \mathbb{N}$, the section $KZ(g_{t+a+jL})(\phi) : [e^{2t}a, e^{2t}b] \rightarrow H^1(M, \mathbb{R})$, is still projectively $C_{a,b}\kappa$ -Lipschitz at $g_{t+a+jL} h_s \omega$. Thus, by [Lemma 4.7](#), there exists $\kappa_0 := \kappa_0(a, b, \delta, L)$ such that, for all $\kappa \in (0, \kappa_0)$ if $t \geq t_0$, and $g_{t+a+jL} h_s \omega \in \hat{U}$, then

$$\text{Leb}(\{s' \in [s - e^{-2(t+a+jL)}, s + e^{-2(t+a+jL)}] : |KZ(g_{t+a+(j+1)L}, h_s \omega) \phi(s)| \\ < e^{\lambda L(1-\epsilon)} |KZ(g_{t+a+jL}, h_s \omega) \phi(s)|\}) < \delta e^{-2(t+a+jL)}.$$

Thus,

$$\text{Leb}(\{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : g_{t+a+jL} h_s \omega \in \hat{U} \text{ but} \\ |KZ(g_{t+a+(j+1)L}, h_s \omega) \phi(s)| < e^{\lambda L(1-\epsilon)} |KZ(g_{t+a+jL}, h_s \omega) \phi(s)|\}) < 4\delta e^{-2t}.$$

It then follows that

$$\int_{-e^{-2t}}^{e^{-2t}} \sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} \chi_j(s_0 + s) ds \leq \sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} \int_{-e^{-2t}}^{e^{-2t}} \chi_j(s_0 + s) ds \leq 4 \lfloor \frac{T}{L} \rfloor \delta e^{-2t},$$

which finally implies that

$$\text{Leb}(\{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : \sum_{j=0}^{\lfloor \frac{T}{L} \rfloor - 1} \chi_j(s_0 + s) > 2\sqrt{\delta} \lfloor \frac{T}{L} \rfloor\}) \leq 2\sqrt{\delta} e^{-2t},$$

as claimed.

Choosing $T_3 > 0$ so large that $\frac{L}{T_3} < \epsilon + \delta$ gives a set of measure at least $[1 - (\sqrt{3(\epsilon + \delta)} + 2\sqrt{\delta})]e^{-2t}$, where we have that, whenever $T > T_3$, for at least $\frac{T}{L}(1 - \sqrt{3(\epsilon + \delta)} - 2\sqrt{\delta})$ of our indices the cocycle grows by at least $e^{\lambda L(1-\epsilon)}$. By [Lemma 2.1](#), the cocycle reduces the Hodge norm by multiplication times a factor larger than e^{-L} on the remaining indices, thereby giving the bound. $\square$

4.2. *Some probabilistic results.* For the reader's convenience, we collect below some well-known probabilistic results.

LEMMA 4.9. *Let (Ω, μ) be a probability space and $F_i : (\Omega, \mu) \rightarrow \{0, 1\}$ be a sequence of random variables such that there exists $0 < \rho < 1$ so that for any j , the conditional probability that F_j is 1 given $F_1, \dots, F_{j-1}$ is at least ρ . Let $G_i : (\Omega, \mu) \rightarrow \{0, 1\}$ be independent and such that $\mu(G_i^{-1}(1)) = \rho$. Then for all ℓ and r ,*

$$\mu \left(\left\{ \omega : \sum_{i=1}^{\ell} F_i(\omega) \leq r \right\} \right) \leq \mu \left(\left\{ \omega : \sum_{i=1}^{\ell} G_i(\omega) \leq r \right\} \right).$$

By standard large deviations results we obtain:

COROLLARY 4.10. *Let (Ω, μ) be a probability space and $F_j : (\Omega, \mu) \rightarrow \{0, 1\}$ be a sequence of random variables such that there exists $0 < \rho < 1$ so that for any j , the conditional probability that F_j is 1 given $F_1, \dots, F_{j-1}$ is at least ρ . For all $\epsilon > 0$, there exists $C_1, C_2 > 0$ so that*

$$\mu \left(\left\{ \omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell \right\} \right) \leq C_1 e^{-C_2 \ell}.$$

COROLLARY 4.11. *Let (Ω, μ) be a probability space, $k \in \mathbb{N}$ and $F_j : (\Omega, \mu) \rightarrow \{0, 1\}$ be a sequence of random variables such that there exists $0 < \rho < 1$ so that for any j , the conditional probability that F_j is 1 given $F_1, \dots, F_{j-k}$ is at least ρ . For all $\epsilon > 0$ there exists $C_3, C_4 > 0$ so that*

$$\mu \left(\left\{ \omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell \right\} \right) \leq C_3 e^{-C_4 \ell}.$$

Proof. Let us consider the k sequences of random variables: $(F_{1+jk})_{j \in \mathbb{N}}$, $(F_{2+jk})_{j \in \mathbb{N}}, \dots, (F_{k+jk})_{j \in \mathbb{N}}$. For each $i \in \{1, \dots, k\}$, the sequence $(F_{i+jk})_{j \in \mathbb{N}}$ satisfies the hypothesis of [Corollary 4.10](#). In fact, by assumption the conditional probability that F_{i+jk} is 1 given $F_1, \dots, F_{i+jk-k}$ is at least ρ . Since $\{F_i, \dots, F_{i+(j-1)k}\} \subset \{F_1, \dots, F_{i+jk-k}\}$, it follows that the conditional probability that F_{i+jk} is 1 given $F_{i+1}, \dots, F_{i+(j-1)k}$ is also at least ρ . By [Corollary 4.10](#) we therefore have that for each $i \in \{1, \dots, k\}$,

$$\mu \left(\left\{ \omega : \sum_{j=1}^{\ell} F_{i+jk}(\omega) \leq (\rho - \epsilon)\ell \right\} \right) \leq C_1(\epsilon) e^{-C_2(\epsilon)\ell}.$$

Finally we have

$$\left\{ \omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell \right\} \subset \bigcup_{i=1}^k \left\{ \omega : \sum_{j=1}^{\lfloor \frac{\ell-i}{k} \rfloor} F_{i+jk}(\omega) \leq (\rho - \epsilon)\ell/k \right\},$$

hence there exists $\ell_0 := \ell_0(k, \epsilon)$ such that for $\ell \geq \ell_0$, we have

$$\begin{aligned}
 (13) \quad & \mu \left(\left\{ \omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell \right\} \right) \\
 & \leq \sum_{i=1}^k \mu \left(\left\{ \omega : \sum_{j=1}^{\lfloor \frac{\ell-i}{k} \rfloor} F_{i+jk}(\omega) \leq (\rho - \epsilon/2) \lfloor \frac{\ell-i}{k} \rfloor \right\} \right) \\
 & \leq kC_1(\epsilon/2)e^{-C_2(\epsilon/2)(\ell/k-2)}.
 \end{aligned}$$

Thus the estimate in the statement holds (for ℓ sufficiently large) with $C_3(\epsilon) = kC_1(\epsilon/2)e^{2C_2(\epsilon/2)}$ and $C_4(\epsilon) = C_2(\epsilon/2)$. $\square$

5. The key proposition

We recall that our strategy, based on Veech's criterion for weak mixing (see [Section 6.1](#) below), consists in establishing, on a large measure set of parameters, growth of the distance $\|\cdot\|_{\mathbb{Z}}$ from the integer lattice of the images of the cohomology classes of the line $\mathbb{R} \cdot [\Im(\omega)]$ under the Kontsevich–Zorich cocycle at arbitrarily large return times of the Teichmüller orbit $\{g_t\omega | t > 0\}$ to a given compact subset $\mathcal{K}$ of the moduli space. The following statement is the key step in our argument.

PROPOSITION 5.1. *Let κ_0 and $\mathcal{F} \subset \hat{\mathcal{F}}$ be as in [Lemma 4.8](#). There exist constants $\sigma > 0, \tau > 0$ so that for all large enough compact sets $\mathcal{K}$, there exists $\gamma_{\mathcal{K}} > 0$ such that all horocycle sections $\phi : [-1, 1] \rightarrow \mathcal{F} \subset \hat{\mathcal{F}}$ at ω , under the conditions that, for some $s_0 \in [-1, 1]$ we have*

(a) *e^t large enough and $\gamma \in (0, \gamma_{\mathcal{K}})$;*

(b) *$g_t h_{s_0} \omega \in \mathcal{K}$,*

(c) *the section $\phi_{t,s_0} : [-1, 1] \rightarrow H^1(M, \mathbb{R})$ defined by*

$$(14) \quad \phi_{t,s_0}(s) = KZ(g_t, h_{s_0+se^{-2t}}\omega)\phi(s_0 + se^{-2t})$$

is $\gamma\kappa$ -Lipschitz at $h_{s_0}\omega$, with $\kappa \in (0, \min\{1, \kappa_0\})$,

(d) $\max_{s \in [-1, 1]} \|\phi_{t,s_0}(s)\|_{\mathbb{Z}} = \gamma$,

there exists a set¹ $\mathcal{S}_{t,s_0} \subset [s_0 - e^{-2t}, s_0 + e^{-2t}]$ of Lebesgue measure

$$(15) \quad \text{Leb}(\mathcal{S}_{t,s_0}) > 2(1 - \gamma^{\sigma})e^{-2t}$$

such that, for each $s \in \mathcal{S}_{t,s_0}$, there exists $\ell := \ell(s)$ such that

(A) $\frac{1}{2}|\log \gamma| \leq \ell \leq \frac{3}{4}|\log \gamma|$;

(B) $g_{t+\ell} h_s \omega \in \mathcal{K}$;

(C) $\max_{s' \in [-1, 1]} \|\phi_{t+\ell,s}(s')\|_{\mathbb{Z}} \geq \|\phi_{t+\ell,s}(0)\|_{\mathbb{Z}} > e^{\tau\ell}(1 - \kappa)\gamma$;

¹Note that these are the parameters that (c) refers to.

The proposition formally implies the following corollary, with different choices of constants. Although the constants in the corollary may be different from those in the proposition, we denote them with the same symbol for ease of notation².

COROLLARY 5.2. *Let κ_0 be as in Lemma 4.8. Let $\tilde{\mathcal{F}}$ be a direct sum of continuously varying strongly irreducible subbundles, each with a positive exponent. There exist constants $\sigma > 0, \tau > 0$ so that for all large enough compact sets $\mathcal{K}$, there exists $\gamma_{\mathcal{K}} > 0$ such that all horocycle sections $\phi: [-1, 1] \rightarrow \hat{\mathcal{F}}$ at ω , under the conditions that, for some $s_0 \in [-1, 1]$, we have*

- (a) e^t large enough and $\gamma \in (0, \gamma_{\mathcal{K}})$;
 - (b) $g_t h_{s_0} \omega \in \mathcal{K}$,
 - (c) the section $\phi_{t,s_0}: [-1, 1] \rightarrow H^1(M, \mathbb{R})$ defined by
- $$(16) \quad \phi_{t,s_0}(s) = KZ(g_t, h_{s_0 + se^{-2t}} \omega) \phi(s_0 + se^{-2t})$$

is $\gamma_{\mathcal{K}}$ -Lipschitz at $h_{s_0} \omega$, with $\kappa \in (0, \min\{1, \kappa_0\})$,

- (d) $\max_{s \in [-1, 1]} \|\phi_{t,s_0}(s)\|_{\mathbb{Z}} = \gamma$,
- there exists a set $\mathcal{S}_{t,s_0} \subset [s_0 - e^{-2t}, s_0 + e^{-2t}]$ of Lebesgue measure*

$$(17) \quad \text{Leb}(\mathcal{S}_{t,s_0}) > 2(1 - \gamma^{\sigma})e^{-2t}$$

such that, for each $s \in \mathcal{S}_{t,s_0}$, there exists $\ell := \ell(s)$ such that

- (A) $\frac{1}{2} |\log \gamma| \leq \ell \leq \frac{3}{4} |\log \gamma|$;
- (B) $g_{t+\ell} h_s \omega \in \mathcal{K}$;
- (C) $\max_{s' \in [-1, 1]} \|\phi_{t+\ell,s}(s')\|_{\mathbb{Z}} \geq \|\phi_{t+\ell,s}(0)\|_{\mathbb{Z}} > e^{\tau \ell} (1 - \kappa) \gamma$;

Note that by our assumption on $\ell(s_0)$, conclusion (C) implies that there exists $0 < \rho < 1$ so that for all $s \in \mathcal{S}_{t,s_0}$ and for $\ell = \ell(s)$, we have

$$(18) \quad \|KZ(g_{\ell}, g_t h_s \omega) \phi(s)\|_{\mathbb{Z}} > (1 - \kappa) \frac{\gamma^{\rho}}{\gamma} \|KZ(g_t, h_s \omega) \phi(s)\|_{\mathbb{Z}} \\ \geq (1 - \kappa)^2 (\|KZ(g_t, h_s \omega) \phi(s)\|_{\mathbb{Z}})^{\rho}.$$

Before beginning the proof of Proposition 5.1, we reduce lower estimates on the distance to the integer lattice to lower bounds on the norm of vectors in the cohomology bundle over a fixed compact set.

²In particular, the bounds on the number of subbundles and on the angle between them alter the estimates in formulas (15) and (C) by multiplicative constants. However, given any σ', τ' strictly smaller than the minimum over the possible subbundles $\mathcal{F}$ of the constants σ, τ given in the previous proposition, if $\gamma_{\mathcal{K}}$ is small enough, we can absorb the multiplicative constants. The basic fact is that if c, ϵ, C are fixed and $0 < x < 1$ is small enough, then $Cx^{c+\epsilon} < x^c < \frac{1}{C}x^{c-\epsilon}$.

LEMMA 5.3. *For any compact subset $\mathcal{K}$ of the moduli space there exists a constant $\gamma'_\mathcal{K} > 0$ such that for $0 < \gamma < \gamma'_\mathcal{K}$ the following holds: Let $\phi: [-1, 1] \rightarrow H^1(M, \mathbb{R})$ be a γ -Lipschitz horocycle section at ω . If $g_\ell \omega \in \mathcal{K}$ and $\max_{s \in [-1, 1]} \|\phi(s)\|_{\mathbb{Z}} \leq \gamma$, then there exists a parallel section $z(s) \in H^1_{h_s \omega}(M, \mathbb{Z})$ such that, for all $\ell \leq 3|\log \gamma|/4$ and for all $s \in [-1, 1]$, we have*

$$\|KZ(g_\ell, h_s \omega) \phi(s)\|_{\mathbb{Z}} = |KZ(g_\ell, h_s \omega)(\phi(s) - z(s))|.$$

Proof. We remark that over any compact set $\mathcal{K}$ the Hodge length of the shortest vector of the integer lattice $H^1(M, \mathbb{Z})$ has a positive minimum $\delta_\mathcal{K} > 0$. We are assuming that $\max \|\phi(s)\|_{\mathbb{Z}} \leq \gamma$, hence there exists $z \in H^1_\omega(M, \mathbb{Z})$ such that

$$|\phi(0) - z| = \max_{s \in [-1, 1]} \|\phi(s)\|_{\mathbb{Z}} \leq \gamma.$$

Let $z(s) \in H^1_{h_s \omega}(M, \mathbb{Z})$ denote the section given by the parallel transport of $z \in H^1_{h_{s_0} \omega}(M, \mathbb{Z})$, that is, the section such that $\pi_\omega(s)z(s) = z$ for all $s \in [-1, 1]$. Similarly, let $\bar{\phi}(s) \in H^1_{h_s \omega}(M, \mathbb{R})$ denote the parallel transport of the vector $\phi(0)$, that is, the section $\bar{\phi}(s)$ such that $\pi_\omega(s)\bar{\phi}(s) = \phi(0)$, for all $s \in [-1, 1]$.

Since for all $s \in [-1, 1]$ the vector $\bar{\phi}(s) - z(s)$ is obtained from the vector $\phi(s_0) - z$ by parallel transport along a horocycle of length at most 2, for all $s \in [-1, 1]$ we have

$$|\bar{\phi}(s) - z(s)| \leq 10\gamma,$$

and since by hypothesis ϕ is a γ -Lipschitz section it follows that

$$|\phi(s) - \bar{\phi}(s)| \leq 10\gamma.$$

We remark that, since $\ell \leq |\log \gamma|/2$, we have that $\|KZ(g_\ell, \omega)\| \leq \gamma^{-3/4}$. Hence

$$|KZ(g_\ell, h_s \omega)(\phi(s) - z(s))| \leq 20\gamma^{1/4}.$$

Thus it suffices to choose $\gamma'_\mathcal{K}$ so that $(\gamma'_\mathcal{K})^{1/4} < \delta_\mathcal{K}/40$. $\square$

As a consequence, under the hypotheses of the above lemma, a lower bound on $|KZ(g_\ell, h_s \omega)(\phi(s) - z(s))|$ is equivalent to a lower bound on $\|KZ(g_\ell, h_s \omega)\phi(s)\|_{\mathbb{Z}}$, and up to replacing $\phi(s)$ with $\phi(s) - z(s)$ we can estimate $|KZ(g_\ell, h_s \omega)\phi(s)|$ from below.

Proposition 5.1 follows from the next lemma, whose proof we defer until after the proposition's proof.

For any $\epsilon, \delta > 0$, let $\eta := \eta(\epsilon, \delta) > 0$ and $\mu := \mu(\epsilon, \delta) > 0$ be the constants defined in formula (9).

LEMMA 5.4. *Let $T_0 > 0$ be sufficiently large and such that $e^{T_0} \in \mathbb{N}$ and let $T > T_0$. Let $\kappa \in (0, \kappa_0)$ and let $\psi: [-1, 1] \rightarrow H^1(M, \mathbb{R})$ be a horocycle section at $\omega \in \mathcal{K}$ such that ψ is projectively κ -Lipschitz. For all $j \in \{0, \dots, \lfloor \frac{T}{T_0} \rfloor - 1\}$, for $s \in [-1, 1]$, let*

$$\Phi_{jT_0}(s) := KZ(g_{jT_0}, h_s \omega) \psi(s).$$

If $\epsilon, \delta > 0$ are small enough, there exists $\sigma' > 0$ such that

$$(19) \quad \text{Leb} \left(\left\{ s \in [-1, 1] : \# \left\{ j \in \left\{ 0, \dots, \lfloor \frac{T}{T_0} \rfloor - 1 \right\} : \right. \right. \right. \\ \left. \left. \left. |\Phi_{(j+1)T_0}(s)| \geq e^{(\mu-\epsilon)T_0} |\Phi_{jT_0}(s)| \right\} \right\} \right) \\ > (1 - 16\eta) \lfloor \frac{T}{T_0} \rfloor \Bigg) > 2(1 - e^{-\sigma'T}).$$

Remark 5.5. Let $\mathcal{F} \subset \hat{\mathcal{F}}$ be an equivariant subbundle, where the Kontsevich–Zorich cocycle acts strongly irreducibly with a positive exponent, and let λ be the largest such exponent. Let $v : [-1, 1] \rightarrow \mathcal{F}$ be a parallel horocycle section at ω . By the above lemma and [Lemma 2.1](#) one has that for any $\epsilon' > 0$ there exists $c > 0$ and $T_1 \in \mathbb{R}$ so that for all $T > T_1$ we have

$$\lambda(\{s \in [-1, 1] : |KZ(g_T, h_s\omega)v(s)| < e^{(\lambda-\epsilon')T}\}) < e^{-cT}.$$

More generally, one can prove such a large deviation result for any Lipschitz horocycle section $\psi : [-1, 1] \rightarrow \mathcal{F}$. Indeed, by [Proposition 3.12](#) (applying it to a function $f_\emptyset$), there exists a compact set $\mathcal{K}$ and a constant c so that the measure of the set of $s \in [-1, 1]$ so that $|\{0 \leq t \leq T : g_t h_s \omega \notin \mathcal{K}\}| < cT$ decays exponentially with T . (Indeed one can choose ρ so big that if $g_t \omega \in \mathfrak{C}_{\rho_0}$, then $g_{t+\tau} \omega \in \mathfrak{C}_\rho = \mathcal{K}$ for all $0 \leq \tau \leq t_0$.) By [Lemma 4.6](#), the transported horocycle section $KZ(g_T)(\psi)$ at such an $s \in [-1, 1]$ is κ_T -projectively Lipschitz with a constant κ_T which decays exponentially with T .

Proof of Proposition 5.1 assuming Lemma 5.4. For all $c, \gamma > 0$, let $\mathcal{B} := \mathcal{B}(c, \gamma)$ denote the set of $s \in [-1, 1]$ such that for all $\ell \in [\frac{1}{2}|\log \gamma|, (\frac{1}{2}+c)|\log \gamma|]$, we have $g_\ell h_s g_t h_{s_0} \omega \notin \mathcal{K}$. We first prove that for any $c > 0$, and for $\gamma > 0$ sufficiently small, the Lebesgue measure of $\mathcal{B}$ is polynomially small in $\gamma > 0$.

To prove an upper bound on the measure of $\mathcal{B}$, we apply [Corollary 3.11](#) with $S = \frac{1}{2}|\log \gamma|$ and $T := c|\log \gamma|$ sufficiently large to derive that there exists $\zeta \in (0, 1)$ such that

$$\text{Leb}(\mathcal{B}) \leq \zeta^{c|\log \gamma|} = \gamma^{c|\log \zeta|}.$$

Let $\psi = \phi_{t, s_0}$. By the hypotheses of the proposition, the horocycle section ϕ_{t, s_0} at $g_t h_{s_0} \omega$, defined in formula (14), satisfies the hypotheses of [Lemma 5.4](#). Indeed, by the hypotheses (c) and (d) of the Proposition, the section ϕ_{t, s_0} is κ -projectively Lipschitz at $h_{s_0} \omega$ with $\kappa \in (0, \kappa_0)$.

Let $T_0 > 0$ be sufficiently large and such that $e^{T_0} \in \mathbb{N}$ and let $\gamma > 0$ be such that $T := \frac{1}{2}|\log \gamma| > T_0$. As in [Lemma 5.4](#), we partition the interval $[0, \frac{1}{2}|\log \gamma|]$ into $R := \lfloor \frac{|\log \gamma|}{2} \frac{1}{T_0} \rfloor$ equal intervals of length $T_0 > 0$ and an additional interval of length at most T_0 . Let $\mathcal{G}$ be the set of [formula \(19\)](#) in the statement of

[Lemma 5.4](#) applied to the horocycle section ϕ_{t,s_0} , and let $\mathcal{S}_{t,s_0}$ be defined as

$$\mathcal{S}_{t,s_0} := \{s_0 + s' e^{-2t} \mid s' \in \mathcal{G} \setminus \mathcal{B}\}.$$

[Lemma 5.4](#) and the previous paragraph establish the measure lower bound of formula (15), for any $\sigma < \min(\sigma', c|\log \zeta|)$.

Let us assume that $s \in \mathcal{S}_{t,s_0} \subset [s_0 - e^{-2t}, s_0 + e^{2t}]$, hence by definition $e^{2t}(s - s_0) \notin \mathcal{B}$. By the above definition of the set $\mathcal{B}$, there exists

$$\ell_s \in \left[\frac{1}{2} |\log \gamma|, \left(\frac{1}{2} + c \right) |\log \gamma| \right]$$

so that $g_{t+\ell_s} h_s \omega = g_{\ell_s} h_{e^{2t}(s-s_0)} g_t h_{s_0} \omega \in \mathcal{K}$.

Clearly [conditions \(A\)](#) and [\(B\)](#) are satisfied for $\ell := \ell_s$. It remains to verify [condition \(C\)](#). Since $e^{2t}(s - s_0) \in \mathcal{G}$, by [formula \(19\)](#) and [Lemma 2.1](#), there exists $\mu' := \mu - 16\eta - \epsilon > 0$ such that, for all $s \in [-1, 1]$,

$$\begin{aligned} |KZ(g_{RT_0}, g_t h_s \omega) \phi(s)| &= |KZ(g_{RT_0}, h_{e^{2t}(s-s_0)} g_t h_{s_0} \omega) \phi_{t,s_0}(e^{2t}(s - s_0))| \\ &= |\Phi_{RT_0}(e^{2t}(s - s_0))| \geq e^{\mu' T_0 R} |\phi_{t,s_0}(e^{2t}(s - s_0))| = e^{\mu' T_0 R} |\phi(s)|, \end{aligned}$$

and because the Hodge norm can change by a factor of at most $e^{\pm(c|\log \gamma| + T_0)}$ from time $RT_0 = \lfloor \frac{|\log \gamma|}{2} \frac{1}{T_0} \rfloor T_0$ to time $\ell_s \leq (\frac{1}{2} + c)|\log \gamma|$, for $c > 0$ sufficiently small there exists $\tau > 0$ such that

$$|KZ(g_\ell, g_t h_s \omega) \phi(s)| \geq e^{\tau \ell_s} |\phi_{t,s_0}(e^{2t}(s - s_0))|.$$

Since, by our assumption on s , $g_\ell h_{e^{2t}(s-s_0)} g_t h_{s_0} \omega \in \mathcal{K}$, by [\(c\)](#) the horocycle section ϕ_{t,s_0} is γ -Lipschitz at $h_{e^{2t}(s-s_0)} g_t h_{s_0} \omega$ and by [\(d\)](#)

$$(20) \quad \max_{s' \in [-1, 1]} \|\phi_{t,s_0}(s')\|_{\mathbb{Z}} = \gamma.$$

By [Lemma 5.3](#), there exists a parallel section $z_{t,s_0} : [-1, 1] \rightarrow H^1(M, \mathbb{Z})$ such that $z_{t,s_0}(s') \in H_{h_{s'} g_t h_{s_0} \omega}^1(M, \mathbb{Z})$ with

$$(21) \quad \|KZ(g_\ell, h_{s'} g_t h_{s_0} \omega) \phi_{t,s_0}(s')\|_{\mathbb{Z}} = |KZ(g_\ell, h_{s'} g_t h_{s_0} \omega)(\phi_{t,s_0}(s') - z_{t,s_0}(s'))|.$$

In fact, we can apply [Lemma 5.3](#) to the section ϕ_{t,s_0} to get a parallel integer section $z_{t,s_0} : [-1, 1] \rightarrow H^1(M, \mathbb{Z})$ at $g_t h_{s_0} \omega$ such that the above identity holds. By applying the above argument to the curve $\phi_{t,s_0} - z_{t,s_0}$, we therefore conclude that, for $s \in \mathcal{S}_{t,s_0}$,

$$\|\phi_{t+\ell_s, s}(0)\|_{\mathbb{Z}} = \|KZ(g_\ell, g_t h_s \omega) \phi_{t,s_0}(s)\|_{\mathbb{Z}} \geq e^{\tau \ell_s} \|\phi_{t,s_0}(s)\|_{\mathbb{Z}}.$$

Finally, since by hypothesis the section ϕ_{t,s_0} is $\kappa\gamma$ -Lipschitz and by [formula \(20\)](#), we have the lower bound

$$\|\phi_{t+\ell_s, s}(0)\|_{\mathbb{Z}} \geq \min_{s' \in [-1, 1]} \|\phi_{t,s_0}(s')\|_{\mathbb{Z}} \geq (1 - \kappa)\gamma,$$

which completes the argument. $\square$

Proof of Lemma 5.4. Let us recall that by definition, for $s \in [-1, 1]$,

$$\Phi_{jT_0}(s) := KZ(g_{jT_0}, h_s\omega)\phi(s).$$

We say that the pair (j, s) is *good* if $g_{jT_0}h_s\omega \in V_\epsilon$. Observe that if (j, s) is good, then, since by hypothesis the horocycle section ϕ is projectively κ -Lipschitz, by Lemma 4.8 for sufficiently large $T_0 > 0$, we have

$$(22) \quad \text{Leb}\left(\left\{s' \in [s - e^{-2jT_0}, s + e^{-2jT_0}] : |\Phi_{(j+1)T_0, s_0}(s')| < e^{\mu T_0} |\Phi_{jT_0, s_0}(s')|\right\}\right) < \eta e^{-2jT_0}.$$

We now wish to use this estimate and Corollary 4.11 to complete the proof of the lemma. To satisfy the assumptions of Corollary 4.11, we define a sequence of nested partitions of the interval $[-1, 1]$:

$$\mathcal{P}^{(j)} = \{[ie^{-2jT_0}, (i+1)e^{-2jT_0}]\}_{i=-e^{2jT_0}-1}^{e^{2jT_0}-1}.$$

Let $P^{(j)}(s)$ be the unique element of $\mathcal{P}^{(j)}$ such that $s \in P^{(j)}(s)$ and $P_i^{(j)} = [ie^{-2jT_0}, (i+1)e^{-2jT_0}]$.

We claim that by the remark preceding the statement of Lemma 5.4, if $g_{jT_0}h_s\omega \in \mathcal{K}$, then, for any $s' \in P^{(j+1)}(s)$,

$$(23) \quad (1 + 2e^{-2T_0})^{-1} \leq \frac{|\Phi_{jT_0}(s)|}{|\Phi_{jT_0}(s')|} \leq (1 + 2e^{-2T_0}).$$

In fact, since the Teichmüller distance between $g_{jT_0}h_s\omega$ and $g_{jT_0}h_{s'}\omega$ is at most e^{-2T_0} , there exists $g \in SL(2, \mathbb{R})$ at a hyperbolic distance at most e^{-2T_0} from the identity in $SL(2, \mathbb{R})/SO(2)$ such that, for any cohomology class $v \in H_{h_{s'}\omega}^1(M, \mathbb{R})$, we have

$$KZ(g_{jT_0}, h_s\omega)(\pi_{h_s\omega}(v)) = KZ(g, g_{jT_0}h_{s'}\omega)KZ(g_{jT_0}, h_{s'}\omega)(v).$$

By Lemma 2.1, it follows that

$$|KZ(g_{jT_0}, h_s\omega)\pi_{h_s\omega}(v)| \leq \exp(e^{-T_0})|KZ(g_{jT_0}, h_{s'}\omega)(v)|,$$

so that for $v = \phi(s')$ we derive the estimate

$$\frac{|KZ(g_{jT_0}, h_s\omega)\pi_{h_s\omega}(\phi(s'))|}{|KZ(g_{jT_0}, h_{s'}\omega)\phi(s')|} \leq \exp(e^{-2T_0}).$$

Let $K > 0$ denote the Lipschitz constant of ϕ at ω . Since the section ϕ is κ -projectively Lipschitz at ω , there exists a constant $C > 0$ such that it is CK -Lipschitz and $C\kappa$ -projectively Lipschitz at $h_s\omega$. By Lemma 2.1, we then have

$$(24) \quad \begin{aligned} |KZ(g_{jT_0}, h_{s'}\omega)\phi(s')| &\geq e^{-jT_0}|\phi(s')| \\ &\geq e^{-jT_0}(|\phi(s)| - CK) \geq e^{-jT_0}K(\kappa^{-1} - C). \end{aligned}$$

Since $\|KZ(g_{jT_0}, h_s\omega)\| \leq e^{jT_0}$ and $|s - s'| \leq e^{-2(j+1)T_0}$ for $s' \in P^{(j+1)}(s)$, we also have

$$|KZ(g_{jT_0}, h_s\omega)(\phi(s) - \pi_{h_s\omega}(\phi(s')))| \leq CKe^{-2(j+1)T_0},$$

so that we have derived the estimate

$$\frac{\left|KZ(g_{jT_0}, h_s\omega)(\phi(s) - \pi_{h_s\omega}(\phi(s')))\right|}{|KZ(g_{jT_0}, h_{s'}\omega)\phi(s')|} \leq \frac{C\kappa e^{-2T_0}}{1 - C\kappa}.$$

The upper bound in formula (23) then follows from the above estimates, for $\kappa \in (0, \kappa_0)$ and for $T_0 > 0$ sufficiently large. The lower bound follows by symmetry, hence the claim is proved.

Now assume T_0 is large enough so that $e^{-\epsilon T_0} < (1 + e^{-2T_0})^{-1}$ and let

$$B_j = \left\{s : \exists s' \in P^{(j)}(s) \text{ with } \frac{|\Phi_{(j+1)T_0}(s')|}{|\Phi_{jT_0}(s')|} \leq e^{(\mu-\epsilon)T_0} \leq e^{\mu T_0} (1 + 2e^{-2T_0})^{-1}\right\},$$

let $\mathcal{I}_j(s) = \{k < j - 1 : s \in B_k\}$ and

$$\mathcal{G}_j = \{s : \exists s' \in P^{(j+1)}(s) \text{ with } g_{jT_0}h_{s'}\omega \in V_\epsilon\}.$$

Observe that (22) and (23) give

$$(25) \quad \begin{aligned} \text{Leb}(\{s \notin B_j : s \in \mathcal{G}_j \text{ and } \mathcal{I}_j(s) = \vec{v}\}) \\ > (1 - 2\eta) \text{Leb}(\{s \in \mathcal{G}_j : \mathcal{I}_j(s) = \vec{v}\}). \end{aligned}$$

Indeed, by definition $s, s' \in P_k^{(j)}$ for some k implies $\mathcal{I}_j(s) = \mathcal{I}_j(s')$ and (22) and (23) gives the measure estimate conditioned to $s \in \mathcal{G}_j$ (which is equivalent to $s' \in \mathcal{G}_j$).

This lower bound on the conditional probability provides the large deviations estimate via Corollary 4.11. Indeed, by applying the corollary to the sequence of random variables F_j equal, for all $j \in \mathbb{N}$, to the characteristic functions of the sets of $s \notin B_j \cap \mathcal{G}_j$, we have that in the complement of a set of measure exponentially small in $\frac{T}{T_0}$,

$$(26) \quad \#\left\{0 \leq j \leq \left\lfloor \frac{T}{T_0} \right\rfloor : s \notin B_j \text{ or } s \notin \mathcal{G}_j\right\} > (1 - 8\eta) \left\lfloor \frac{T}{T_0} \right\rfloor$$

(for all large enough $T > T_0$). This is because by formula (25) the sequence of random variables $\{F_j\}$ defined above satisfies the assumptions of Corollary 4.11. Consider the set of $s \in [-1, 1]$ such that

$$(27) \quad \left|\left\{j \in \left\{0, \dots, \left\lfloor \frac{T}{T_0} \right\rfloor\right\} : g_{jT_0}h_s\omega \in V_\epsilon\right\}\right| > (1 - 4\eta) \left\lfloor \frac{T}{T_0} \right\rfloor.$$

Note that by Corollary 3.13 (iv) (which we may apply if T_0 is at least t_0) the complement of this set has that its measure decays exponentially with $\left\lfloor \frac{T}{T_0} \right\rfloor$. The intersection of the sets given by (26) and (27) satisfies the desired conditions, hence the proof of the Proposition is complete. $\square$

6. Proof of Theorem 1.3

6.1. *The Veech criterion.* To prove Theorem 1.3 we need a condition to rule out the flow $F_{r_\theta\omega}^t$ having $\alpha \in \mathbb{R}$ as an eigenvalue. This is Lemma 6.1 below, which is essentially due to Veech. We remark that the second named author has an alternate version of the criterion, where one has a weaker assumption (only one time in the compact set instead of three) and obtains (essentially³) the same conclusion [For24, Thm. 4.12]. Following Veech [Vee84, §7], we have the following criterion for a translation flow $F_{r_\theta\omega}^t$ to not have α as an eigenvalue. There is a $c > 0$ and a sequence of transversals $\{J_i\}$ so that

- $|J_i| \rightarrow 0$.
- $F_{r_\theta}^s(J_i)$ are disjoint intervals for all $0 \leq s \leq \frac{c}{\ell'(J_i)}$, where $\ell'(J_i)$ denotes the length of the projection of J_i in the direction θ .
- If T is the interval exchange transformation given by the first return to J_i and $I_1, \dots, I_d$ are the intervals that define T , then $|I_a| > c|I_b|$ for all a, b .

Let r_i be the vector of return times of $F_{r_\theta\omega}^t$ to J_i . If

$$\limsup_{i \rightarrow \infty} \|r_i \alpha\|_{\mathbb{Z}} \neq 0,$$

then α is not an eigenvalue of $F_{r_\theta\omega}^t$. We have the following trivial consequence of this:

LEMMA 6.1. (Veech) *Let α be a nontrivial eigenvalue of F_ω^t . For every compact set $\mathcal{K}$, there exists $\ell_{\mathcal{K}}$ so that, if there exist $\ell \geq \ell_{\mathcal{K}}$ and a diverging sequence $(t_i) \subset \mathbb{R}^+$ with the property that $g_{t_i}\omega \in \mathcal{K}$, $g_{t_i-\ell}\omega \in \mathcal{K}$ and $g_{t_i+\ell}\omega \in \mathcal{K}$, then*

$$\lim_{i \rightarrow \infty} \|KZ(g_{t_i}, \omega)(\alpha \Im(\omega))\|_{\mathbb{Z}} = 0.$$

Proof. For each compact set $\mathcal{K}$, there exist constants $a_{\mathcal{K}}$, $b_{\mathcal{K}}$ and $d_{\mathcal{K}} > 0$ so that, if $\omega' \in \mathcal{K}$, then we can write ω' as a zippered rectangles over an interval I of length at most $a_{\mathcal{K}}$, in the horizontal direction and one endpoint a singularity (that is, as a suspension with piece-wise constant roof function). Moreover, the heights of the rectangles are at most $b_{\mathcal{K}}$. Lastly $d_{\mathcal{K}}$ is the length of the shortest saddle connection on all $\omega' \in \mathcal{K}$. Because there are singularities on each endpoint of the rectangles, if u is the width of a rectangle, there is a saddle connection on the surfaces with holonomy (u, r) with $r < b_{\mathcal{K}}$. So if $\ell > \log(\frac{b_{\mathcal{K}}}{2d_{\mathcal{K}}})$ and $g_\ell\omega' \in \mathcal{K}$, then $u \geq \frac{1}{2}d_{\mathcal{K}}e^{-\ell}$. In fact, the holonomy of the

³The result of the second author assumes that the eigenvalue has no continuous eigenfunction; the case when the eigenvalue has a continuous eigenfunction has an easier proof following Veech's argument.

same saddle connection on $g_\ell\omega$ is $(e^\ell u, e^{-\ell} r)$, and for $\ell > \log(\frac{2b_K}{d_K})$ we have

$$e^{-\ell}|r| \leq \frac{d_K}{2b_K} b_K = d_K/2.$$

Since $g_\ell\omega' \in \mathcal{K}$, its shortest saddle connection has length at least d_K , hence we have $e^\ell|u| \geq d_K/2$, as claimed.

Likewise, because one endpoint of I is a singularity, if $F_{\omega'}^s$ is discontinuous on I , then there is a saddle connection on ω' with holonomy (a, v) with $|v| \leq |s|$ and $|a| < a_K$. Thus, if $\ell > \log(\frac{a_K}{2d_K})$ and $g_{-\ell}\omega' \in \mathcal{K}$, then $F_{\omega'}^s$ acts continuously on I for all $-\frac{1}{2}e^{-\ell}d_K < s < \frac{1}{2}e^{-\ell}d_K$. In fact, the saddle connection on $g_{-\ell}\omega'$ has holonomy $(e^{-\ell}a, e^\ell v)$. For $\ell > \log(\frac{2a_K}{d_K})$ we have

$$e^{-\ell}|a| \leq \frac{d_K}{2a_K} a_K = d_K/2,$$

hence $e^\ell|v| \geq d_K/2$ and so $|s| \geq d_K/2e^{-\ell}$, as claimed.

We choose $\ell_K = \max\{\log(\frac{a_K}{2d_K}), \log(\frac{b_K}{2d_K})\}$.

Now, if $\omega' = g_{t_i}\omega$, by g_{-t_i} we can transport I back to ω and obtain an interval I' so that $|I'| = \ell'(I') \leq e^{-t_i}a_K$. Observe that the second and third bullet points of the Veech criterion still hold since they state properties which are equivariant with respect to the action of the Teichmüller flow g_t . We now need to understand the return times to I' . Note that they are the imaginary parts of a basis for (absolute) homology of the surface and are all of bounded size when transported to $g_{t_i}\omega$ (because $g_{t_i}\omega = \omega' \in \mathcal{K}$). Thus the return time vector is given by $KZ(g_{t_i}\omega)\Im(\omega)$, and the lemma follows. $\square$

6.2. Eliminating the stable subbundle of the $SL(2, \mathbb{R})$ -bundle and isometric subbundles. Recall the decomposition of r_θ given by [formula \(4\)](#). Since the horocycle $\hat{h}_t$ fixes the real part of holomorphic differentials, that is, it fixes the leaves of the stable foliation of the Teichmüller flow, we have that $g_{\log \cos \theta} h_{-\tan \theta} \omega$ belongs to the same stable manifold as $r_\theta \omega$, hence $h_{-\tan \theta} \omega$ and $r_\theta \omega$ have the same vertical foliation. The vertical flow of $h_{-\tan \theta} \omega$ equals the vertical flow of $r_\theta \omega$ after a linear reparametrization by multiplication times $\cos \theta$. It follows that if α is a (fixed) eigenvalue for the vertical flow of $r_\theta \omega$, then $\alpha \sec(\theta)$ is an eigenvalue for the vertical flow of $h_{-\tan \theta} \omega$. Thus, by the Veech criterion ([Lemma 6.1](#)) we have that, if $\mathcal{K}$ is a compact set, $\ell \geq \ell_K$, $g_{t_i} h_s \omega, g_{t_i - \ell} h_s \omega$ and $g_{t_i + \ell} h_s \omega \in \mathcal{K}$ with $t_i \rightarrow \infty$, then

$$\lim_{i \rightarrow \infty} \|KZ(g_{t_i}, h_s \omega) \alpha \sec(\theta) [\Im(h_{-\tan \theta} \omega)]\|_{\mathbb{Z}} \rightarrow 0.$$

We conclude that if α is a fixed eigenvalue for the vertical flow of $r_\theta \omega$ for a positive measure set of $\theta \in \mathbb{T}$, then $\alpha \sec(\arctan s)$ is an eigenvalue for the vertical flow of $h_s \omega$, for a positive measure set of $s \in \mathbb{R}$, which implies that for a

positive measure set of $s \in \mathbb{R}$ we have that if $g_{t_i}h_s\omega, g_{t_i-\ell}h_s\omega$ and $g_{t_i+\ell}h_s\omega \in \mathcal{K}$ with $t_i \rightarrow \infty$, then

$$\lim_{i \rightarrow \infty} \|KZ(g_{t_i}, h_s\omega)\psi(s)\|_{\mathbb{Z}} \rightarrow 0,$$

where $\psi(s)$ is the horocycle section

$$\psi(s) := \alpha \sec(\arctan s) [\mathfrak{S}(h_s\omega)] = \alpha \sec(\arctan s) [\mathfrak{S}(\omega)].$$

LEMMA 6.2. *The section ψ has range in $SL(2, \mathbb{R})$ subspace at ω , and it is transverse to any integer translate of the stable subbundle of the $SL(2, \mathbb{R})$ subbundle.*

Proof. The stable subspace of the $SL(2, \mathbb{R})$ subspace at $h_s\omega$ is generated by the cohomology class $[\mathfrak{R}(h_s\omega)] \in H_{h_s\omega}^1(M, \mathbb{R})$. For any $z \in H_{\omega}^1(M, \mathbb{Z})$, we consider the equation (with $a \in \mathbb{R}$)

$$\psi(s) - z = a\mathfrak{R}(h_s\omega) = a(\mathfrak{R}(\omega) + s\mathfrak{S}(\omega)).$$

By definition of $\psi(s)$ we derive

$$\begin{aligned} z \wedge \mathfrak{S}(\omega) &= -a\mathfrak{R}(\omega) \wedge \mathfrak{S}(\omega), \\ z \wedge \mathfrak{R}(\omega) &= (as - \alpha \sec(\arctan s))\mathfrak{R}(\omega) \wedge \mathfrak{S}(\omega). \end{aligned}$$

Thus the coefficient $a \in \mathbb{R}$ is uniquely determined (given $z \in H_{\omega}^1(M, \mathbb{Z})$) by the first equation, hence the second equation has a unique solution. $\square$

PROPOSITION 6.3. *For every compact set $\mathcal{K}$ there exists a constant $\gamma''_{\mathcal{K}} > 0$ such that the following holds: If there exist $t_0 > 0$ and an integer class $z \in H_{g_{t_0}\omega}^1(M, \mathbb{Z})$ such that*

- $g_{t_0}\omega \in \mathcal{K}$ and $\{t > 0 | g_t\omega \in \mathcal{K}\}$ has upper density at least $\frac{3}{4}$;
- $|KZ(g_{t_0}, \omega)(\alpha[\mathfrak{S}(\omega)]) - z| \in (0, \gamma''_{\mathcal{K}})$,⁴
- $KZ(g_{t_0}, \omega)(\alpha[\mathfrak{S}(\omega)]) - z$ belongs to an isometric subbundle E_0 for the Kontsevich–Zorich cocycle,

then α is not an eigenvalue for the vertical flow F_{ω}^t of ω .

Proof of Proposition 6.3. Let $\gamma''_{\mathcal{K}}$ be so small that the minimum Hodge distance between points of the lattice $H_{\omega}^1(M, \mathbb{Z})$ for $\omega \in \mathcal{K}$ is at least $10\gamma''_{\mathcal{K}}$. By the isometry property of the Kontsevich–Zorich cocycle on E_0 , we have that for all $t \geq t_0$,

$$\begin{aligned} (28) \quad \|KZ(g_{t_0}, \omega)(\alpha[\mathfrak{S}(\omega)])\|_{\mathbb{Z}} &= |KZ(g_{t_0}, \omega)(\alpha[\mathfrak{S}(\omega)]) - z| \\ &= |KZ(g_t, \omega)(\alpha[\mathfrak{S}(\omega)]) - KZ(g_{t-t_0}, g_{t_0}\omega)(z)| \leq \gamma''_{\mathcal{K}}, \end{aligned}$$

⁴Note that this condition cannot hold if $\alpha = 0$.

hence, whenever $g_t\omega \in \mathcal{K}$, since $KZ(g_{t-t_0}, g_{t_0}\omega)(z) \in H^1_{g_t\omega}(M, \mathbb{Z})$, by the choice of the constant $\gamma_{\mathcal{K}} > 0$, we have

$$\|KZ(g_t, \omega)(\alpha[\mathfrak{S}(\omega)])\|_{\mathbb{Z}} = \|KZ(g_{t_0}, \omega)(\alpha[\mathfrak{S}(\omega)])\|_{\mathbb{Z}} \neq 0.$$

Since, by hypothesis that the set $\{t > 0 : g_t\omega \in \mathcal{K}\}$ has upper density $\frac{3}{4}$, it can be proved (by an elementary argument left to the reader) that for any $\ell_0 > 0$ there exist $\ell > \ell_0$ and a diverging sequence (t_i) such that $g_{t_i}\omega$, $g_{t_i-\ell}\omega$ and $g_{t_i+\ell}\omega \in \mathcal{K}$ for all $i \in \mathbb{N}$. Hence, we may apply the Veech criterion (Lemma 6.1), and α is not an eigenvalue, as stated. $\square$

6.3. *Proof of Theorem 1.3.* Before completing the proof, we recall that, if the Kontsevich–Zorich cocycle does not act isometrically on a subbundle, then Corollary 5.2 can be applied to it.

THEOREM 6.4. (cf. [EM18], [FMZ14], [Fil17]) *The Hodge bundle (hence the symplectic orthogonal of the $SL(2, \mathbb{R})$ subbundle) splits as a sum of an $SL(2, \mathbb{R})$ -invariant continuous (in fact, algebraic) subbundle, on which the Kontsevich–Zorich cocycle acts isometrically, and a finite number of $SL(2, \mathbb{R})$ -invariant continuous (algebraic) irreducible subbundles on each of which the cocycle has a strictly positive exponent. After passing to a finite cover, the irreducible components of the decomposition can be taken to be strongly irreducible, in the sense that they do not admit a measurable almost invariant splitting.*

Proof. By [EM18, Thm. A.6] the Kontsevich–Zorich cocycle on Hodge bundle is semi-simple, in the sense that it can be split as a direct sum of $SL(2, \mathbb{R})$ -invariant measurable subbundles, and each irreducible component is either isotropic or symplectic. By [EM18, Thm. A.5] if the subbundle is isotropic then all of the exponents are zero. By [EM18, Thm. A.4], if all the exponents are zero then the cocycle acts isometrically. It follows that the Hodge bundle can be split as a sum of an isometric component, equal to the sum of all isotropic irreducible components, and a finite sum of symplectic irreducible components with at least one non-zero exponent. Each symplectic non-isometric component has both a positive and a negative exponent (which are opposite of each other). After passing to a measurable finite cover, we can assume that each irreducible component is strongly irreducible. By [Fil17, Thms. 1.4 and 1.5], the splitting of the Hodge bundle into irreducible components is continuous, in fact real algebraic, hence the irreducible components are strongly irreducible after passing to an algebraic finite cover. $\square$

COROLLARY 6.5. *Let ϕ be a horocycle section. For almost every $s \in \mathbb{R}$, one of the following two (mutually exclusive) possibilities holds:*

- (1) $\phi(s)$ belongs to a continuous $SL(2, \mathbb{R})$ -equivariant subbundle where the Kontsevich–Zorich cocycle acts isometrically;

- (2) $\phi(s)$ has a non-trivial projection onto a continuous $SL(2, \mathbb{R})$ -equivariant subbundle where the Kontsevich–Zorich cocycle has a positive exponent and acts strongly irreducibly.

Proof of Theorem 1.3. We proceed with a proof by contradiction. Let $B \subset [-1, 1]$ denote the set of points for which the conclusion of the theorem fails, and assume that the measure of B is $\xi > 0$. Let $\mathcal{K}$ be a compact set so that Lemma 2.2 and Corollary 5.2 hold with $c = .999$ and $\mu(Int(\mathcal{K})) > .999$. Let $a = \max\{t_0, \ell_{\mathcal{K}}\}$, where t_0 is as in Lemma 2.2 and $\ell_{\mathcal{K}}$ is as in Lemma 6.1. We choose $\gamma_0 > 0$ small enough so that Corollary 5.2 holds with the compact set $\mathcal{K}$, $\frac{1}{2}|\log(\gamma_0)| > T_0$, where T_0 is as in Section 5, $\gamma_0 \leq \gamma_{\mathcal{K}}''$ where $\gamma_{\mathcal{K}}''$ is as in Proposition 6.3, and two additional smallness conditions below (so that formula (33) holds and the right-hand side in the estimate of formula (35) below is at most $\frac{1}{2}$). By Lemma 6.1, there exists $t_0 > 0$ so that

$$(29) \quad \text{Leb}(\{s \in [-1, 1] : \text{for all } t \geq t_0 \text{ either } \|KZ(g_t, h_s\omega)\psi(s)\|_{\mathbb{Z}} < e^{-a-1}\gamma_0 \\ \text{or } g_t h_s\omega \notin \mathcal{K}, \text{ or } g_{t-a} h_s\omega \notin \mathcal{K}, \text{ or } g_{t+a} h_s\omega \notin \mathcal{K}\}) > .99\xi.$$

Let B' denote the set in the left-hand side of formula (29). By the Lebesgue density theorem, there exists $r_0 > 0$ so that

$$(30) \quad \text{Leb}(\{s \in B' : \text{Leb}(B(s, r) \cap B') > 1.99r \text{ for all } r \leq r_0\}) > .98\xi$$

Let B'' be the set in the left-hand side of formula (30). We now choose $t \geq t_0$ and $s \in B''$ such that $h_s\omega$ is Birkhoff generic, which is a full measure condition by [CE15, Thm. 1.1], so that

- $e^{-2t} < r_0$
- $g_t h_s\omega \in \mathcal{K}$
- $\|KZ(g_t, h_s\omega)\psi(s)\|_{\mathbb{Z}} = \gamma < \gamma_0$.

The first assumption is trivial. The second and third assumptions can be simultaneously satisfied by Lemma 6.1 and Birkhoff genericity. (Indeed $\mu(\mathcal{K}) > .999 > \frac{3}{4}$ and so, for any ℓ , there exist arbitrarily large t so that $g_{t-\ell} h_s\omega, g_t h_s\omega, g_{t+\ell} h_s\omega \in \mathcal{K}$.)

By the third, and last, of the above assumptions there exists an integer class $z \in H_{g_t h_s\omega}^1(M, \mathbb{Z})$ such that

$$|KZ(g_t, h_s\omega)(\psi(s) - z)| = \gamma < \gamma_0.$$

We now show that if $\psi(s) - z$ is contained in an isometric subbundle, then for almost every $\theta \in S^1$, the flow in direction θ does not have α as an eigenvalue. Indeed, we are assuming all but the first assumption of Proposition 6.3, and that assumption holds for almost every $s \in [-1, 1]$, by Birkhoff genericity.

Thus, up to the translation of ψ by the integer vector z , that is, by considering the section $\psi - z$ instead of ψ , we can assume that ψ has a non-zero

projection on a strongly irreducible subbundle with a non-zero top Lyapunov exponent.

By [Lemma 6.2](#), the section ψ is transverse to any integer translate of the stable subbundle of the $SL(2, \mathbb{R})$ subbundle, so clearly the projection to the stable subspace of the $SL(2, \mathbb{R})$ subbundle can be ignored.

To complete the proof, we now separately consider the projection of ϕ to the unstable part of the $SL(2, \mathbb{R})$ subbundle and to the invariant subbundle $\hat{\mathcal{F}}^+ \subset \hat{\mathcal{F}}$, given by the sum of all strongly irreducible subbundles with a positive exponent in the symplectic orthogonal $\hat{\mathcal{F}}$ of the $SL(2, \mathbb{R})$ subbundle.

We show that for each of these individually, at the time $t' > t$ when the norm of the section has grown to size γ_0 under the action of the cocycle, the subset of $B(s, e^{-2t})$ of points that are not in the set in formula (29) has at least half the measure of $B(s, e^{-2t})$, a contradiction, which will conclude our argument. First, we consider the projection onto the unstable part of the $SL(2, \mathbb{R})$ -subbundle. By [Lemma 2.2](#), for any $t' > t_0 + a$ we have that

$$(31) \quad \text{Leb}(\{s' \in B(s, e^{-2t}) : g_{t'}h_{s'}\omega, g_{t'-a}h_{s'}\omega \text{ and } g_{t'+a}h_{s'}\omega \in \mathcal{K}\}) > 1.994e^{-2t}.$$

Let $\delta > 0$ be the norm of the projection of $KZ(g_t, h_s\omega)\psi(s)$ onto the unstable part of the $SL(2, \mathbb{R})$ subbundle. Let $\ell = \log(\frac{\gamma_0}{\delta}) + 1$. Because the unstable part of the $SL(2, \mathbb{R})$ subbundle grows by a constant factor $e^{t'}$ at time $t' > 0$, we have our claim by applying (31) at $t' = t + \ell$.

It now suffices to prove the analogous result for $\hat{\mathcal{F}} = \hat{\mathcal{F}}^+ \oplus \mathcal{I}$, where $\hat{\mathcal{F}}^+$ is, as above, the sum of all subbundles where the cocycle acts strongly irreducibly with a positive exponent and $\mathcal{I}$, where it acts isometrically. Such a decomposition is guaranteed by [Theorem 6.4](#). Let ϕ be the projection of ψ on $\hat{\mathcal{F}}^+$. By choosing a possibly larger $t > 0$, we can further assume that

- $g_t h_s \omega \in \mathcal{K}$,
- the horocycle section $\phi_0 : [-1, 1] \rightarrow H^1(M, \mathbb{R})$ defined by

$$\phi_0(s') = KZ(g_t, h_{s+e^{-2t}s'}\omega)\phi(s + e^{-2t}s')$$

is $\kappa\gamma'$ -Lipschitz at $h_s\omega$, where $\kappa < \kappa_0$ as in [Corollary 5.2](#),

- $\max_{s' \in [-1, 1]} \|\phi_0(s')\|_{\mathbb{Z}} = \gamma' < \gamma_0$.

Indeed, by [Lemma 4.6](#), for every compact set $\mathcal{K}'$, there exists $v > 0$ so that to establish the second bullet point, it suffices to show that $|\{0 < t'' < t : g_{t''}h_{s'}\omega \in \mathcal{K}'\}| > v > 0$ for all $s' \in [s - e^{-2t}, s + e^{-2t}]$. We choose $\mathcal{K}'$ to be the closure of a small neighborhood of $\mathcal{K}$. By the Birkhoff genericity of $h_s\omega$ with respect to the Teichmüller flow, there exists t_1 so that $|\{0 < t'' < t_1 : g_{t''}h_s\omega \in \mathcal{K}\}| > v > 0$. There exists U , a small neighborhood of s , so that for every $s' \in U$ and $0 \leq t'' \leq t_1$,

$$g_{t''}h_{s'}\omega \in \mathcal{K}' \text{ whenever } g_{t''}h_s\omega \in \mathcal{K}.$$

The second bullet point follows by choosing $t \geq t_1$ so that

$$s' \in [s - e^{-2t}, s + e^{-2t}] \subset U.$$

We now iteratively apply [Corollary 5.2](#). For every $j \in \mathbb{N}$, let $\mathcal{S}_j \subset [s - e^{-2t}, s + e^{2t}]$ denote the set of points such that [Corollary 5.2](#) can be applied j -times. Inductively, by applying [Corollary 5.2](#) for $j - 1$ times, for a point $s_0 \in \mathcal{S}_j$ we obtain real numbers $\ell_1(s_0), \dots, \ell_j(s_0)$. Let $L_j(s_0) := \ell_1(s_0) + \dots + \ell_j(s_0)$ and, for all $s' \in [-1, 1]$, let

$$\phi_j(s') := KZ(g_{t+L_j(s_0)}, h_{s'}\omega)\phi(s'),$$

which restricts on the interval $[s_0 - e^{-2(t+L_j(s_0))}, s_0 + e^{-2(t+L_j(s_0))}]$ to a reparametrization of the section $\phi_{t+L_j(s_0), s_0}$ as in [Corollary 5.2 \(C\)](#). Since the section $\phi_{t+L_j(s_0), s_0}$ is $\gamma'\kappa$ -Lipschitz and has maximum at least γ' , it follows that

$$\min \phi_{t+L_j(s_0), s_0} \geq (1 - \kappa) \max \phi_{t+L_j(s_0), s_0}.$$

Let then (x_j) denote the sequence defined by recursion as follows: let $x_0 = \min \phi_0$ and let

$$x_{j+1} = (1 - \kappa)^2 x_j^\rho.$$

Notice that, for all $j \in \mathbb{N}$,

$$x_j = (1 - \kappa)^{2+\dots+2\rho^j} x_0^{\rho^j} \geq (1 - \kappa)^{2/(1-\rho)} x_0^{\rho^j}.$$

Observe that, if

$$(32) \quad \beta := \log_{x_0}(\gamma_0) - 2(1 - \rho)^{-1} \log_{x_0}(1 - \kappa) > 0$$

and $v = \lceil \log_\rho(\beta) \rceil$, then, if γ_0 is small enough, we have

$$(33) \quad 2\gamma_0^\rho > x_v \geq \gamma_0.$$

Indeed, we have $x_0^{\rho^{\log_\rho(\beta)}} = \gamma_0(1 - \kappa)^{-\frac{2}{1-\rho}}$. Now, by induction, if $s' \in \mathcal{S}_{j+1}$ and $\|KZ(g_{t+L_i(s')}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} < \gamma\kappa$ for $i \leq j$, then

$$(34) \quad \|KZ(g_{t+L_{j+1}(s')}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} \geq x_{j+1}.$$

Indeed, inductively $x_i \leq \|KZ(g_{t+L_i(s')}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}}$, and so it follows by [formula \(18\)](#). To see this, we now check that, if $s' \in \mathcal{S}_j$, then we can apply [Corollary 5.2](#) at $h_{s'}\omega$ if and only if $\|KZ(g_{t+L_j(s')}, h_{s'}\omega)\phi(s')\|$ is small enough. First, $e^{t+L_j(s')} > e^t$ and $g_{t+L_j(s')}h_{s'}\omega \in \mathcal{K}$ by the definition of $\mathcal{S}_j$. Second, the Lipschitz assumption follows because [Lemma 4.2](#) gives that the Lipschitz constant gets smaller, while at the same time $\|KZ(g_{t+L_j(s')}, h_{s'}\omega)\phi(s')\| \geq \|KZ(g_t, h_{s'}\omega)\phi(s')\|$, and so the requirement on the Lipschitz constant gets laxer. So the second bullet point implies [\(c\)](#).

We now bound from below the measure of the set of these points to obtain a contradiction with [\(30\)](#), thereby completing the proof. By [\(17\)](#), we

have that the the measure of the set of $s' \in [s - e^{-2t}, s + e^{-2t}]$ so that $\|KZ(g_{t+L_{\min\{j:s' \notin S_j\}-1}(s'), h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} < \gamma_0$ is at most (since $v = \lceil \log_\rho(\beta) \rceil$ and by [formula \(32\)](#))

$$(35) \quad \sum_{j=0}^v (x_j)^\sigma \leq \sum_{j=0}^v (x_0^{\rho^j})^\sigma \leq \sum_{i \geq -1} (x_0^{\beta^\sigma})^{\frac{1}{\rho^i}} = \sum_{i \geq 0} \left(\frac{\gamma_0}{(1 - \kappa)^{2/(1-\rho)}} \right)^{\rho^\sigma \frac{1}{\rho^i}}.$$

Clearly, this bound goes to 0 with γ_0 , and so, if γ_0 is small enough, we are left with at least a subset of conditional measure greater than $\frac{1}{2}$. That is, for a set of $s' \in [s - e^{-2t}, s + e^{-2t}]$ of conditional measure at least $\frac{1}{2}$, we have that for each s' in this set

- there exists $\tau_{s'} \geq t$ so that $\|KZ(g_{\tau_{s'}}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} \geq \gamma_0$,
- $g_{\tau_{s'}}h_{s'}\omega \in \mathcal{K}$.

By [Lemma 2.2](#), if γ_0 is sufficiently small (so that in [Corollary 5.2](#) $\ell(s) \geq |\log \gamma_0|/2 > t_0$, hence $\tau_{s'} - t \geq t_0$), the subset of $s' \in [s - e^{-2t}, s + e^{-2t}]$ such that $g_{\tau_{s'}+a}h_{s'}\omega$ or $g_{\tau_{s'}+2a}h_{s'}\omega$ are not in $\mathcal{K}$ has conditional measure at most $1/9$. By [Lemma 2.1](#), we have that

$$\|KZ(g_{\tau_{s'}}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} \geq \gamma_0 \implies \|KZ(g_{\tau_{s'}+a}, h_{s'}\omega)\phi(s')\|_{\mathbb{Z}} \geq e^{-a}\gamma_0.$$

By the above lower bound, together with the condition that $g_{\tau_{s'}}h_{s'}\omega \in \mathcal{K}$, $g_{\tau_{s'}+a}h_{s'}\omega \in \mathcal{K}$ and $g_{\tau_{s'}+2a}h_{s'}\omega \in \mathcal{K}$, we conclude that $s' \notin B'$. Since the conditional measure of such $s' \in [s - e^{-2t}, s + e^{-2t}]$ is at least $1/2 - 1/9 > 1/4$, this contradicts that $s \in B''$ (because $e^{-2t} < r_0$). $\square$

7. Proof of [Theorem 1.1](#)

7.1. Setup. Let P be a polygon and $T_1(P)$ be the phase space for its billiard flow. Let $X(P) = T_1(P) \times T_1(P)$.

The phase space $T_1(P)$ is endowed with the Liouville measure for the billiard flow, hence the space $X(P)$ can be endowed with the product measure. All integrals below will be taken with respect to the square of the Liouville measure on $X(P)$.

We also need to define $\text{Lip}_c(X(P))$, the space of c-Lipschitz functions on $X(P)$. For instance, we view $T_1(P)$ as a quotient of $P \times S^1$ (which is a metric space) and use the path metric to define a metric on $T_1(P)$.

[Theorem 1.1](#) follows readily from the next lemma and the Baire category Theorem. The proof of this lemma will be presented after the proof of [Theorem 1.1](#) assuming the lemma.

LEMMA 7.1. *For every $\epsilon > 0$ and $T > 0$ we have*

$$(36) \quad \left\{ P : \int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon \right. \\ \left. \text{for all } f \in \text{Lip}_1(X(P)) \right\} \text{ is open.}$$

By Theorem 1.2 we have

COROLLARY 7.2. *For every $\epsilon > 0$, there exists N_ϵ such that the following holds: If P is a rational polygon with the property that the group of reflections about its sides (translated to the origin) contains a rotation by $\frac{2\pi}{M}$ with $M \geq N_\epsilon$, then there exists T_0 such that for all $T \geq T_0$ and $f \in \text{Lip}_1(X(P))$, we have*

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon.$$

Proof. Consider the flat surface M_P obtained by unfolding P . By Theorem 1.2, we have that for almost every pair of directions the product flow is ergodic on $M_P \times M_P$. Considering these flows on $T_1(P)$, by our assumption on the group of reflections, they equidistribute in the product of the table cross $M \geq N_\epsilon$ evenly spaced copies of a discrete set in S^1 . The corollary follows. $\square$

Proof of Theorem 1.1 assuming Lemma 7.1. Let n be given and, for all $k \in \mathbb{N}$, let U_k be the set of polygons P with $n \geq 3$ sides so that there exists $T > 0$ such that, for all $f \in \text{Lip}_1(X(P))$,

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \frac{1}{k}.$$

By Lemma 7.1, U_k is open for all $k \in \mathbb{N}$ as union (over $T > 0$) of a family of open sets. By Corollary 7.2 it is dense for all $k \in \mathbb{N}$. So by the Baire Category theorem there exists $P \in \cap_k U_k$. Because 1-Lipschitz functions have dense span in the set of continuous functions, the product of the billiard flow with itself on $X(P)$ is ergodic. In fact, if $P \in \cap_k U_k$ we have that, for every $k \in \mathbb{N}$, there exists $T_k > 0$ such that for all $f \in \text{Lip}_1(X(P))$,

$$\int_{X(P)} \left| \frac{1}{T_k} \int_0^{T_k} f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \frac{1}{k}.$$

Ergodicity then follows (for instance) from the ergodic theorem. The ergodicity of the product billiard flow implies that the billiard flow is weakly mixing. $\square$

Definition 7.3. (Y. Vorobets [Vor97, Def. 2.1]) We say a polygon Q is a δ -perturbation of P if there exists a homeomorphism $\phi_{P,Q}: P \rightarrow Q$ that establishes a one-to-one correspondence between the vertices of the two polygons and so that the distance between the corresponding vertices is at most δ .

For every polygon P , there exists $\delta_P > 0$ such that for $\delta < \delta_P$, any δ -perturbation Q of P has a triangulation whose triangles are in a bijective correspondence with triangles of a triangulation of P . For instance, one can triangulate P by diagonals in an arbitrary manner. Then for $\delta < d(P)/2$, with $d(P)$ the minimum of the smallest non-zero distance between vertices, for each two vertices of P which can be joined by a diagonal inside P , the corresponding vertices of Q can also be joined by a diagonal inside P . Then there exists a triangulation of Q corresponding to the above triangulation of P by diagonals.

In the next lemma we assume that the homeomorphism $\phi_{P,Q}$ is as above, affine on the triangles in a (fixed) triangulation of P and takes them to the corresponding triangles of Q .

LEMMA 7.4. *For every table P and for every $\epsilon > 0$, there exists $\delta > 0$ such that if Q is a δ -perturbation of P , then*

$$\text{Lip}_{1-\epsilon}(X(Q)) \subset \phi_{P,Q}^{-1}(\text{Lip}_1(X(P))).$$

This is straightforward.

LEMMA 7.5. *If P is in the set described by (36), then there exists $\epsilon' > 0$ so that*

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon$$

for all $f \in \text{Lip}_{1+\epsilon'}(X(P))$.

Proof. The table P is compact and $\text{Lip}_1(X(P))$ is bounded and equicontinuous, so by the Arzelà–Ascoli Theorem, $\text{Lip}_1(X(P))$ is compact in $\|\cdot\|_{\text{sup}}$. Thus, there exists $0 < \epsilon'' < \frac{1}{9}$ so that

$$(37) \quad \int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon - \epsilon'',$$

for all $f \in \text{Lip}_1(X(P))$.

Indeed, for each $f \in \text{Lip}_1(X(P))$, there exists ϵ_f so that

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon - \epsilon_f.$$

Thus, there is an open neighborhood U_f of f that, for all $h \in U_f$,

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T h(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} h \right| < \epsilon - \frac{1}{2}\epsilon_f.$$

The claim in formula (37) follows by compactness.

Let then $\epsilon' = \frac{1}{9}(\frac{1}{1-\epsilon''} - 1)$ and $h \in \text{Lip}_{1+\epsilon'}(X(P))$. Now from (37) applied to $f = h/(1 + \epsilon')$ we derive the following bound:

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T h(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} h \right| < (1 + \epsilon')(\epsilon - \epsilon'') < \epsilon.$$

The last inequality uses that $\epsilon'\epsilon < (1 + \epsilon')\epsilon''$. $\square$

For any polygon P , let $\pi^{(1)} : P \times S^1 \rightarrow P$ and $\pi^{(2)} : P \times S^1 \rightarrow S^1$ denote the canonical projections.

PROPOSITION 7.6. (*Vorobets [Vor97, Prop. 2.3]*) *Let Q be a δ -perturbation of P . For each $t > 0$, there exists $B \subset P \times S^1$ dependent on the polygons P, Q , the map $\phi := \phi_{P,Q}$ and t and of measure at most $C_3(C_1t + C_2)^3\delta$ such that for each $(x, \theta) \in P \times S^1 \setminus B$ and for each τ , $0 \leq \tau \leq t$, at least one of the following two possibilities hold:*

- (1) *The distance between $\pi_P^{(1)} F_P^t(x, v)$ and $\phi^{-1}(\pi_Q^{(1)} F_Q^t(\phi(x), v))$ is at most*

$$C_4(C_1t + C_2)^2\delta,$$

and the angle between the directions of $\pi_P^{(2)} F_P^t(x, v)$ and $\pi_Q^{(2)} F_Q^t(\phi(x), v)$ is at most $C_5(C_1t + C_2)\delta$.

- (2) *The points $\pi_P^{(1)} F_P^t(x, v)$ and $\pi_Q^{(1)} F_Q^t(\phi(x), v)$ lie at a distance at most*

$$C_6(C_1t + C_2)^2\delta$$

from the boundaries of P and Q , respectively.

The constants $C_1, \dots, C_6$ are positive and depend only on P .

From this proposition, we derive:

COROLLARY 7.7. *For any $f \in \text{Lip}_c(X(P))$, T and $\epsilon > 0$, there exists $\delta > 0$ so that for any Q , a δ -perturbation of P we have*

$$(38) \quad \int_{X(Q)} \left| \int_0^T f \circ \phi_{P,Q}^{-1}(F^t(x, \theta), F^t(y, \psi)) dt - \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt \right| < \epsilon.$$

Note that for fixed c and T , δ can be chosen uniformly over $\text{Lip}_c(X(P))$

Proof of Lemma 7.1. Let $\epsilon < \frac{1}{4}$. It suffices to show that for any P that belongs to the set in formula (36), there exists $\delta_P > 0$ so that any δ_P -perturbation also belongs to the set (36). Because P belongs to the set (36), it satisfies the inequality in (36) for $\epsilon - \epsilon'''$ for some $\epsilon''' > 0$. We choose $\delta > 0$ so that

[Lemma 7.4](#) implies that, if Q is a δ -perturbation of P , then $\text{Lip}_1(X(Q)) \subset \text{Lip}_{1+\frac{\epsilon'''}{9}}(X(P))$, formula (38) is satisfied with $\epsilon = \frac{\epsilon'''}{9}$ and

$$\left| \int_{X(Q)} f \circ \pi_{P,Q}^{-1} - \int_{X(P)} f \right| < \frac{\epsilon'''}{9}.$$

It follows that any δ -perturbation of P satisfies (36). $\square$

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