CEF: counterexamples and replacement arguments

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This note supplies the arguments behind the public review. It concerns selected assertions in Church–Ellenberg–Farb, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. 164 (2015), 1833–1910. The original paper's FI-sharp classification is retained as an input where explicitly used. Standard bundle, general-position and universal-coefficient facts are stated; this is not a formal verification of their foundations.

1. A manifold counterexample to Theorem 1.9(c)

Set $N = L^7(3)\#UTS^4\#UTS^4$ and $M = N \setminus \{p\}$. All three summands are closed connected oriented seven-manifolds. Deleting a point is diffeomorphic to taking the interior of the compact complement of a coordinate ball, so the boundary hypothesis is satisfied.

The lens space has intermediate homology $\mathbb{Z}/3$ in degrees $1, 3, 5$, and zero in the other intermediate degrees. One way to obtain this is to view it as the circle bundle over $\mathbb{CP}^3$ with Euler class $3h$: its Gysin maps between successive even cohomology groups are multiplication by three, followed by Poincaré duality. The oriented bundle $S^3 \to UTS^4 \to S^4$ has Euler number two. Its Gysin sequence gives middle cohomology $H^4 = \mathbb{Z}/2$, hence middle homology $H_3 = \mathbb{Z}/2$ and no other intermediate homology. Mayer–Vietoris for the connected sum, and then puncturing, give

$$H_1(M) = \mathbb{Z}/3, \quad H_2(M) = 0, \quad H_3(M) = \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2.$$

All positive-degree rational homology of M vanishes. For $X_m = M^m$, the split integral homological Künneth sequence yields

$$H_1(X_m) = (\mathbb{Z}/3)^m, \qquad H_2(X_m) = (\mathbb{Z}/3)^{\binom{m}{2}}.$$

On adjoining one factor, degree-three homology gains $H_3(M)$, $H_2(X_m) \otimes H_1(M)$ and $\mathrm{Tor}_1(H_1(X_m), H_1(M))$. Thus

$$H_3(M^n) = (\mathbb{Z}/2)^{2n} \oplus (\mathbb{Z}/3)^{n+\binom{n}{2}+\binom{n}{3}}.$$

The splitting is a group splitting, not a claim of naturality. Rational Künneth implies $\mathrm{Hom}(H_4(M^n), \mathbb{Z}) = 0$. The cohomological universal-coefficient sequence therefore identifies $H^4(M^n; \mathbb{Z})$ with the Ext of the displayed group, giving the same primary multiplicities.

Each pair diagonal in M^n has codimension seven. A relative chain of dimension below seven can be perturbed, relative to its boundary and compatibly on faces, to miss every pair diagonal. There are only finitely many such submanifolds for a fixed n; their union need not itself be a submanifold. Relative homology vanishes in degrees below seven, so configuration-space inclusion induces homology isomorphisms through degree five. Naturality of the cohomological UCT gives the needed degree-four isomorphism. At $n = 0, 1$ the assertion is immediate.

For a finite abelian group G, reduction at a prime forces at least $\dim_{\mathbb{F}_p}(G/pG)$ generators. Aligning cyclic generators of the different primary factors attains the largest of these counts, by the Chinese remainder theorem. Hence

$$\mu H^4(\mathrm{Conf}_n(M); \mathbb{Z}) = \max\left(2n, \frac{n^3 + 5n}{6}\right).$$

The cubic wins for $n \geq 3$, but not at one or two. The all-size polynomial assertion is false.

2. Quadratics on the reduced square-rank variety

Let $X_n = \{A \in \mathrm{Mat}_n(\mathbb{C}) : \mathrm{rank}(A^2) \leq 1\}$. Write $\Delta_{I,J}$ for a two-by-two minor and $e_2 = \sum_I \Delta_{I,I}$.

Every rank-one matrix belongs to X_n. Substituting $x_{ij} = u_i v_j$ in a vanishing quadratic shows that it is a linear combination of the two-by-two minors: a monomial in the u's and v's has two degree-two preimages precisely when both its row and column indices are distinct, and their difference is a minor.

Every nonprincipal coefficient can be eliminated separately. For disjoint row and column pairs choose a two-arrow matrix sending the column basis vectors to the row basis vectors and killing the others. It squares to zero and has exactly that one nonzero minor. For pairs sharing one index, use a two-arrow chain $b \mapsto c \mapsto a \mapsto 0$. Its square has rank one and again only the desired minor survives.

To compare principal coefficients, conjugate $J = E_{12} + E_{23}$ by $I + tE_{31}$. The resulting matrix $J + tE_{32} - tE_{21}$ has principal minors $t, -t, 0$ on the three coordinate pairs, and its square still has rank one. Coefficients on pairs sharing one index are equal. Such comparisons connect all pairs for $n \geq 3$; for $n = 2$ there is only one pair.

Conversely, the generalized eigenspaces of A for nonzero eigenvalues have total dimension at most one, since A^2 is invertible on their sum. The second elementary function of the eigenvalues is therefore zero. The coefficient identity for $\det(I + tA)$ identifies this with e_2. Thus the quadratic vanishing space is exactly $\mathbb{C}e_2$ for $n \geq 2$.

Equivalently this is the degree-two part of the radical of the minor ideal. Radical membership can be checked after passing modulo each prime over that ideal and then to the algebraic closure of its fraction field; the same rank argument kills e_2. Consequently

$$\dim \mathcal{O}(X_n)^2_{\text{red}} = \binom{n^2+1}{2} - 1 \quad (n \geq 2).$$

For $n = 1$ the correct dimension is one, which differs from the uniquely determined eventual polynomial. The nonreduced minor ideal is generated in degree four and has no degree-two equations; the distinction is essential.

3. A finite, exact disproof for the minor scheme

Let J_n be generated by the two-minors of A^2, and let $g(n) = \dim_{\mathbb{C}}(J_n)_5$. Put $m = \binom{n}{2}$, $W = \bigwedge^2 \mathbb{C}^n$, and $R(A) = \bigwedge^2(A^2)$. The entries of $R(A)$ are precisely the quartic generators. Every degree-five ideal element is a linear combination of variables times these generators. Accordingly, $g(n)$ is the rank of

$$\mu : \operatorname{Hom}(\operatorname{Mat}_n, \operatorname{End} W) \longrightarrow \mathbb{C}[x_{ij}]_5, \qquad L \longmapsto (A \mapsto \operatorname{tr}(L(A)R(A))).$$

Define $D(A)(u \wedge v) = Au \wedge v + u \wedge Av$. Since $D(A)$ commutes with $R(A)$, every fixed $C \in \operatorname{End} W$ gives a relation $L(A) = [D(A), C]$. For $n \geq 2$ these relations have dimension $m^2 - 1$. Indeed, an operator commuting with every $D(A)$ first commutes with all diagonals and so is diagonal in the wedge basis, whose weights $h_i + h_j$ are distinct linear forms. Commuting with the matrix units equates its entries on pairs differing by one index. That graph is connected, including the singleton case $n = 2$. The common centralizer is therefore precisely the scalars. This proves the characteristic-zero upper bound

$$g(n) \leq (n^2 - 1)\binom{n}{2}^2 + 1 \qquad (n \geq 2).$$

For the finite sizes zero through eight, the bundled checker constructs the integer coefficient matrix directly, expanding

$$(A^2)_{ik}(A^2)_{jl} - (A^2)_{il}(A^2)_{jk}$$

and multiplying by each variable. It verifies that $1{,}000{,}003$ is prime and performs exact modular Gaussian elimination. Rank over this field is a lower bound for rank over $\mathbb{Q}$: a nonzero modular minor comes from a nonzero integer determinant. At every tested size it reaches the preceding upper bound. Thus these finite ranks are exact over $\mathbb{Q}$ and $\mathbb{C}$:

$$0,\ 0,\ 4,\ 73,\ 541,\ 2401,\ 7876,\ 21169,\ 49393.$$

For every size, regardless of the sharper upper bound, the elementary generator count gives $0 \leq g(n) \leq n^2\binom{n}{2}^2 = O(n^6)$. If Theorem 5.3.2 held in degree five, then

$$g(n) = \binom{n^2+4}{5} - \dim(\mathbb{C}[x_{ij}]/J_n)_5$$

would be a polynomial, necessarily of degree at most six by that growth bound. The computed values give $\Delta^7 g(0) = -6$ and $\Delta^7 g(1) = 1$. Either contradicts polynomiality. The latter avoids size zero altogether.

This proof does not infer a formula for all sizes from finite samples. It combines finite exact rank computations with a uniform degree bound to disprove the claimed polynomial. The longer diagonal/shear argument in the focused reviewer record addresses a universal rank formula; that formula is unnecessary for this finite disproof. Independent final Sol and Fable reviews accepted its characteristic-zero exhaustion argument, with explicit conjugation, block-projection and first-order expansion clarifications. Those longer derivations are preserved in the reviewer records. The finite proof above needs only the displayed upper bound.

4. Correct integral generator formula

For an integral FI-sharp module with finite generation, write its coefficient groups in the classification as W_j, with free ranks r_j and primary generator counts $t_{j,p}$. Only finitely many W_j and torsion primes occur. The subset decomposition gives

$$R(n) = \sum_j r_j \binom{n}{j}, \qquad T_p(n) = \sum_j t_{j,p} \binom{n}{j}.$$

The exact minimum is $R(n) + \max_p T_p(n)$, with zero for absent torsion. The free generators contribute to every residue-field lower bound; aligning the primary generators attains it. Since a finite collection of polynomial differences has eventually constant signs, one of the T_p controls the maximum for all sufficiently large n. This proves eventual polynomiality with the same degree upper bound. It leaves rational rank and each fixed-primary count polynomial for every size. Mod-p cohomology is not in general the reduction of integral cohomology, and has its own FI-sharp decomposition.

5. A replacement for the general-ring proof

Assume M is connected, oriented, of dimension $d \geq 2$, and the interior of a compact manifold with nonempty boundary. Put $D_i = i$ for $d \geq 3$ and $D_i = 2i$ for $d = 2$. The following proof retains the homotopy FI-sharp construction and its classification. Its other inputs are the ordinary configuration forgetful fibration, the spectral sequence with local coefficients, and the integral universal-coefficient theorem.

Here is the homotopy compatibility needed for both homology and cohomology. Choose a connected boundary collar with depth $0 < t < 1$. Its compression $h_u(t) = u + (1-u)t$ extends by the identity outside the collar. A partial injection first forgets unwanted labels and relabels; when it adds labels, compress by a fixed $u > 0$ and insert their distinct constants at depths below u. With no insertions use $u = 0$. Two compressions compose to h_v with $v = b + (1-b)a$. Retained first-stage constants have depths between b and v, and second-stage constants are below b. All lie in the vacated region. Varying v and rescaling these depths, then moving the ordered constants within that region, gives the direct composite up to homotopy. Its configuration space is path connected in dimension at least two. If no constants survive, isotope the compression to the identity. This also covers points that are added and subsequently forgotten.

Homology is covariant for these homotopy maps. Partial inversion converts contravariant cohomology into the required covariant FI-sharp functor. The subset coefficient extraction is exact: on a finite set S, the commuting idempotents e_s that forget s give mutually orthogonal projectors $\prod_{s \in T}(1 - e_s) \prod_{s \notin T} e_s$. They split each pointwise exact sequence into its subset components. No assertion that the Leray spectral sequence is FI-sharp natural is required.

Forgetting the last point in an ordered configuration gives a bundle over $\mathrm{Conf}_m(M)$ with fiber M minus m labeled points. Local trivializations move points in disjoint coordinate balls. Excision identifies the relative cohomology of this puncturing with F^m in degree d, and zero otherwise, over any field F. Ordered labels and the orientation make this local system constant. The old cohomology of M also has trivial monodromy, since transport extends to an ambient isotopy. The punctured-fiber cohomology has a filtration by constant systems; the extension itself need not be constant.

Let $c_{m,i} = \dim_F H^i(\mathrm{Conf}_m(M); F)$ and $b_b = \dim_F H^b(M; F)$. Long exact sequences for these local systems and the Serre spectral sequence give, for $i > 0$,

$$c_{m+1,i} \le c_{m,i} + \sum_{b=1}^{i} b_b c_{m,i-b} + m c_{m,i-(d-1)}.$$

Negative indices mean zero and $c_{m,0} = 1$. Induction on i bounds the increment by $O((m+1)^{i-1})$ for $d \ge 3$, and by $O((m+1)^{2i-1})$ for $d = 2$. Summation gives $c_{n,i} = O((n+1)^{D_i})$ over each field separately. Uniform constants across all fields are not needed.

Integral homology of M is degreewise finitely generated. In the smooth or PL case one can use finite CW type of the compact model. More generally, compact manifolds are compact ENRs and finitely dominated: a retraction from a Euclidean neighborhood can be restricted to a finite union of small cubes covering the manifold. Finite domination suffices for finite generation of homology. The puncturing pair sequence and homological Serre spectral sequence propagate degreewise finite generation to each configuration space. The fiber systems again have finite filtrations by constant finitely generated groups. This finiteness step matters: field bounds alone would not exclude divisible integral groups.

Write integral homology as $H_j(\mathrm{Conf}_\bullet(M); \mathbb{Z}) = M(W^{(j)})$ using FI-sharp. Each $W_r^{(j)}$ is a finitely generated abelian summand. A nonzero such summand is detected after tensoring with $\mathbb{Q}$ or an appropriate $\mathbb{F}_p$. The homological UCT injects the tensor product into field homology. The $\binom{n}{r}$ copies of the summand and the field growth bound force $r \le D_j$.

Now use the natural exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{i-1}, k) \longrightarrow H^i(\mathrm{Conf}_\bullet(M); k) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_i, k) \longrightarrow 0.$$

It is k-linear and natural in the space. Hom and Ext reverse arrows; partial inversion identifies the resulting contravariant FI-sharp structures with covariant ones. They respect the finite subset sums, projections and relabelings. Extracting the coefficient groups is exact under the FI-sharp/FB equivalence, so support bounds pass through this extension. Both ends have support at most D_i.

For a free cyclic coefficient, Hom gives k and Ext gives zero. For $\mathbb{Z}/h$, they give $k[h] = \mathrm{Ann}_k(h)$ and k/hk. These are finitely generated when k is commutative Noetherian. The middle term is therefore generated in degrees at most D_i. No finite-Krull-dimension hypothesis was used. Over a general non-Noetherian ring the support conclusion remains distinct from finite generation of its annihilator ideals.

This proof never assumes that the Leray spectral sequence is FI-sharp, nor that tensoring an arbitrary inclusion remains injective.

6. Which rank-variety repairs are valid?

For ordinary determinantal conditions $p_\ell(T) = c_\ell T$, a minor pulled back by a partial injection is a signed source minor, multiplied by the scalar power, or zero. The homogeneous minor ideal is compatible, and the same is true of its radical by taking powers. The original FI-sharp decomposition then gives all-size dimension polynomiality. Degree-d monomials use at most $2d$ labels, supplying the degree bound. The fixed-subset trace formula supplies all-size character polynomials in characteristic zero.

For positive monomials $p_\ell(T) = c_\ell T^{s_\ell}$, $s_\ell \ge 1$, zero extension of matrices is still compatible; principal restriction generally is not. Each fixed-degree quotient is co-FI, and its dual U is an FI-submodule of the dual polynomial space.

Here is a direct eventual dimension argument over any field. Restrict to increasing injections of ordered sets. Order the dual monomial basis lexicographically by the exponent arrays in lexicographic pair order. Increasing injections preserve comparisons by inserting zero entries. Leading indices of vectors in U are consequently closed under these injections. At fixed degree, each index has one of finitely many patterns on $q \le 2d$ used labels, and a tuple of $q+1$ nonnegative gaps before, between and after those labels.

For a fixed pattern the allowed gap tuples form an upward set under coordinatewise comparison. It has finitely many minimal tuples: from an infinite sequence in $\mathbb{N}^{q+1}$ one can successively extract a nondecreasing subsequence in every coordinate, contradicting an infinite antichain. Each tuple dominates a minimal one by descent.

For a minimal tuple b, its upper orthant contributes $\binom{n-\sum b_i}{q}$ indices for all sufficiently large n. Intersections use coordinatewise maxima. Inclusion-exclusion therefore gives an eventual polynomial, of degree at most q for that pattern. Summing

patterns proves eventual dimension polynomiality of degree at most $2d$. Elimination identifies the number of leading indices with $\dim U_n$. Choosing a vector for each minimal leading index and cancelling leading terms also proves finite FI generation.

In characteristic zero, the general FI character theorem then gives eventual character polynomiality. The dual has the same character, since a permutation is conjugate to its inverse. This last step is an explicit use of the original foundational theorem, not an independent proof of all of FI theory. None of this restores the lost all-size statement for matrix powers.

Verification and limits

The integer matrix checks, the minor witnesses and the finite-field rank calculation are independently executable from the check bundle. The shipped results record the coordinating assistant's runs; rerunning the scripts is a fresh check. The rank disproof needs only the finite computations and the commutator upper bound in Section 3, not the full universal syzygy classification.

The prior reports supplied the discovery examples. Initial source-only Sol/Fable readings, focused Astra challenges and separate final checks are preserved as distinct review records. The final publication records which arguments were accepted and which subsidiary obligations remain. These checks do not establish priority, absence of earlier corrections, human peer review, or correctness of every external theorem cited by CEF.