Review of FI-modules and stability for representations of symmetric groups

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This AI-assisted, source-adjudicated review identifies false application statements in Church, Ellenberg and Farb's paper, together with proof defects that admit repairs. It does **not** overturn the central theory of FI-modules. The distinction between a polynomial formula for every size and one valid only for sufficiently large sizes is essential to the findings.

The reference edition is the published paper, *Duke Mathematical Journal* **164** (2015), 1833–1910, DOI 10.1215/00127094-3120274. The independent readers received the complete final author version, arXiv:1204.4533v4. We then compared the disputed application passages with the supplied 78-page journal PDF. The journal retains them. This is a targeted comparison, not a claim that the two editions are identical.

The counterexamples and repairs supply the detailed mathematics. The check bundle contains the executable arithmetic checks and recorded results. The neutral reading page and its source map are separate from these findings.

Findings at a glance

Published location	Finding	What survives
Theorem 1.9(c), pp.1838–1839	False: the integral minimum-generator count need not be polynomial for all sizes.	An exact maximum-of-polynomials formula, and eventual polynomiality.
Theorem 5.3.2, p.1888	False in its stated generality, including a homogeneous, nonconstant example for the minor scheme. The general setup also need not be graded.	Ordinary determinantal conditions retain the all-size conclusion; positive-power conditions admit an eventual replacement.
Remark 5.3.3, p.1888	False for the reduced variety defined by rank of the square at most one.	Eventual polynomiality in this example; more general repairs need compatible maps.
Proof of Theorem 6.4.3, pp.1897–1898	A subquotient does not give the claimed inequality after reduction modulo a maximal ideal.	A universal-coefficient proof recovers bounded generation; finite Krull dimension is unnecessary in that proof.
Lemma 6.3.2(b), p.1893	The converse from a filtered module to its associated graded pieces is false.	The forward implication used in the stability argument holds.
Main FI-module structure theorems	No counterexample to the principal finite-generation, Noetherian, classification or eventual-character results was established by this audit.	The scoped positive assessments retain their imported prerequisites and qualifications.

The first three rows are false statements, rather than merely missing explanations. A repaired statement does not make the original quantifiers true.

1. Integral generators: why a maximum appears

For a finitely generated abelian group, the minimum number of generators is its free rank plus the largest generator count among its primary torsion parts. Different primes do not add their requirements: a single generator can serve coprime cyclic factors simultaneously.

An FI-sharp decomposition gives finite binomial sums

$$R(n) = \sum_j r_j \binom{n}{j}, \qquad T_p(n) = \sum_j t_{j,p} \binom{n}{j}.$$

The exact integral minimum is

$$\mu(n) = R(n) + \max_p T_p(n).$$

Each fixed-prime count is polynomial. Their maximum need not be a single polynomial on all nonnegative integers. It is eventually polynomial because only finitely many primes occur and the signs of finitely many polynomial differences eventually stabilize.

This is realized by an actual manifold satisfying the paper's hypotheses. Let

$$M = (L^7(3) \# UTS^4 \# UTS^4) \setminus \{\text{point}\}.$$

Here UTS^4 is the unit tangent sphere bundle of the four-sphere. The space M is a connected oriented seven-manifold, diffeomorphic to the interior of a compact manifold with nonempty boundary. Standard bundle, Künneth and universal-coefficient calculations give

$$H^4(\mathrm{Conf}_n(M); \mathbb{Z}) \cong (\mathbb{Z}/2)^{2n} \oplus (\mathbb{Z}/3)^{n+\binom{n}{2}+\binom{n}{3}}.$$

Removing the pair diagonals does not change these low-degree groups: the diagonals have codimension seven. Therefore

$$\mu(n) = \max\left\{2n, \frac{n^3+5n}{6}\right\}.$$

This equals the cubic for every $n \geq 3$, but at $n = 1, 2$ its values are $2, 4$ rather than $1, 3$. No polynomial can agree with a cubic on an infinite tail and disagree with it at those two inputs. This refutes Theorem 1.9(c). The detailed note supplies the topology and group calculation; an abstract FI-sharp example alone would not have established a counterexample to the manifold theorem.

The rational, fixed-field and fixed-prime assertions are distinct. This counterexample does not refute them. Degree assertions should also be read as upper bounds: vanishing cohomology cannot have the prescribed exact degree.

2. Rank varieties: three separate failures

The paper studies matrices satisfying rank conditions on polynomials in the matrix and claims all-size polynomiality for fixed homogeneous degrees.

First, arbitrary polynomials need not give homogeneous equations. For $p(T) = T - 1$, the rank conditions generally involve constants and different degrees. There is then no inherited standard grading of the asserted kind. One must specify homogeneous data, or formulate a different filtered problem.

Second, the claimed partial-injection maps need not preserve the conditions. Put

$$N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad A = \mathrm{diag}(N, N).$$

Then $A^2 = 0$. Its principal submatrix on indices $1, 3$ is I_2, whose square has rank two. Thus restriction takes a matrix with $\mathrm{rank}(A^2) \leq 1$ outside that rank variety. The FI-sharp construction used in the proof does not exist as asserted.

Third, this is accompanied by false numerical conclusions. The following two tests keep the reduced variety and the minor scheme separate.

The reduced variety

Over $\mathbb{C}$, let $X_n = \{A : \mathrm{rank}(A^2) \leq 1\}$. For $n \geq 2$ its quadratic vanishing ideal is precisely the span of the second elementary characteristic coefficient $e_2(A)$. Consequently,

$$\dim \mathcal{O}(X_n)^2_{\mathrm{red}} = \binom{n^2+1}{2} - 1 \qquad (n \geq 2).$$

At $n = 1$ the variety is the entire affine line, so the dimension is one, whereas the displayed polynomial gives zero. This refutes Remark 5.3.3. The proof checks both the vanishing of e_2 and the absence of other quadratic equations; it is not an inference from sampled matrices.

The minor scheme

Let J_n be the ideal generated by the 2×2 minors of A^2. Its generators have degree four, so the preceding degree-two calculation does not apply to $\mathbb{C}[x_{ij}]/J_n$. In degree five, set $g(n) = \dim(J_n)_5$. We independently constructed the integer coefficient matrices of all variables times all these minors and computed their ranks modulo the verified prime $1{,}000{,}003$.

An explicit commutator family supplies matching characteristic-zero upper bounds. Thus the following are exact characteristic-zero ranks, not merely probable ranks inferred from floating-point computation:

n	0	1	2	3	4	5	6	7	8
$g(n)$	0	0	4	73	541	2401	7876	21169	49393

There are at most $n^2\binom{n}{2}^2 = O(n^6)$ products. If the quotient's degree-five dimension were a polynomial for all n, its difference from the ambient dimension $\binom{n^2+4}{5}$ would make $g(n)$ a polynomial. The growth bound forces its degree to be at most six. But

$$\Delta^7 g(0) = -6, \qquad \Delta^7 g(1) = 1.$$

This is impossible. The second calculation uses only positive sizes, so discarding the empty matrix does not fix the assertion. This gives a homogeneous, nonconstant counterexample to Theorem 5.3.2 itself.

Finite computations generally cannot prove a universal formula. Here they disprove a polynomial claim because an independent degree bound makes a finite contradictory difference decisive. The detailed note explains the rank bounds and the exact checker. A separate proposed formula for every n is not needed for this disproof.

3. Repairs worth retaining

For integral generators, replace the all-size polynomial by the exact maximum formula above and conclude eventual polynomiality. This preserves the useful quantitative bound without asserting an invalid equality.

For rank varieties, ordinary determinantal conditions are preserved by principal restriction and zero extension. Their homogeneous ideals and radicals consequently support the original all-size argument. Positive powers $p_\ell(T) = c_\ell T^{s_\ell}$, with $s_\ell \geq 1$, retain zero-extension compatibility but generally lose restriction compatibility. A direct ordered-monomial argument gives eventual dimension polynomiality, with degree at most $2d$ in grading d; the characteristic-zero character conclusion additionally uses the general character-polynomial theorem.

For Theorem 6.4.3, reducing a submodule modulo a maximal ideal is not left exact. The replacement proof first obtains polynomial field-cohomology bounds from the forgetful configuration-space fibration. Integral homology is degreewise finitely generated. The FI-sharp decomposition and field detection bound the support of its coefficient groups. The natural cohomological universal-coefficient sequence then gives the same support bound over any commutative Noetherian ring. The proof uses finite generation of annihilator ideals, but not finite Krull dimension. Its standard topological premises and the arrow directions are stated explicitly in the repair note.

These are separately attributed reconstructions. They do not retroactively validate the printed residue-field inference or the rank-variety maps.

4. Other qualifications from the broader reading

The filtered-module equivalence in Lemma 6.3.2(b) has a simple counterexample. Take the constant FI-module $V_n = \mathbb{Q}$ and its submodule $W_n = 0$ below a positive threshold N, $W_n = \mathbb{Q}$ above it. The module V has stability type $(0, 0)$; the transition $W_{N-1} \to W_N$ is not surjective. The associated graded pieces need not inherit the bounds. The forward implication from the pieces to the filtered object follows by exact coinvariants and is the direction needed for the spectral-sequence proof. Numerical injectivity-degree bounds also need truncation at zero when the printed expression is negative. These changes do not reduce the stated final rational stable range in the checked argument.

In Proposition 3.3.3 the displayed multiplicity argument cancels a common summand without a finite-multiplicity hypothesis. Infinite cardinal multiplicities cannot be cancelled in that way. The finite-dimensional applications do not encounter this obstruction. The reviewers propose repairs in the unrestricted setting, but this release does not certify those additional proofs.

The Section 7 reviewers identify a non-proper Gysin-map issue in the definition of the tautological classes, a normalization issue for their pullbacks, and low-genus exceptions to an all-genus Torelli identification. They propose compact-universal-curve normalization and restriction to the genus-at-least-three tail. These observations concern proof interfaces; they do not establish false intended eventual-polynomiality theorems. The detailed external Johnson, Malcev and moduli inputs were not supplied to the panel and have not been exhaustively verified here. These subsidiary findings are reported for follow-up, not promoted to cleared repairs.

5. Evidence, coverage and attribution

The initial round comprised four Sol 6.2 high and four Fable 5.2 high responses. Each received the full native author-version TeX and one of four assigned scopes: foundations, algebraic applications, configuration spaces, or moduli/Torelli/Lie applications. The two owner-supplied earlier reports were withheld from that round. Fourteen grouped targets were assigned to each model; this was not a separate audit of every numbered claim or cited reference.

Two focused Astra 6.2 high responses then challenged the proposed counterexamples and repairs using original passages, prior-report pages and journal images. Three final checks completed: Sol and Fable independently examined the degree-five and reduced rank arguments, and Sol checked the configuration-space examples and replacement coefficient proof. Their full records state their source limitations and the small expository additions needed in the longer kernel calculation. The coordinating assistant inspected the arguments and sources, performed the exact checks, and selected the publication findings. Model agreement is evidence of scrutiny, not a proof certificate or human peer review.

Seven initial responses used descriptive source labels rather than the strict identifiers required by the validator. Their complete mathematical text was preserved in explicit metadata-reconciled derivatives. The original validation failures and unavailable-reference declarations remain in the provenance record. No additional inference was used to reformat them.

Earlier Astra and Fable reports supplied for this investigation are the discovery leads for the principal counterexamples. This review credits that provenance and does not claim priority for the mathematics. A scoped literature search is not evidence that no correction has ever appeared elsewhere.

The journal records receipt on 14 July 2013. External results used to evaluate its original proof are subject to that historical cutoff; present-day self-contained repairs are identified separately. Model training cannot be restricted retrospectively to that date.

The public coverage and cost record includes every submitted attempt, actual model identifiers, token usage, validation status and the conservative usage-cost estimate of **$10.544866** under the owner's $25 total authorization. All 13 submitted assessments returned complete mathematical responses. It is not a provider invoice. The report, reading map, separate decision graph and checks are available together so readers can assess the evidence themselves.