

An omission in the Fuchs–Meiri–Sarnak parameter classification

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Finding. The seven displayed families in Theorem 2.6 do not exhaust the primitive integral hyperbolic hypergeometric parameter pairs in dimensions $n > 9$. The degree-13 example below is omitted. This remains an objection after correcting the obvious formula typos. **It does not refute the two thinness families in Theorem 1.**

Reviewed source. Fuchs–Meiri–Sarnak, *Hyperbolic monodromy groups for the hypergeometric equation and Cartan involutions*, JEMS **16** (2014), 1617–1671. The classification is Theorem 2.6, pp.1626–1627, with proof on pp.1627–1630. Publisher PDF; DOI: 10.4171/JEMS/471.

The explicit pair

Let Φ_m be the complete m th cyclotomic polynomial. Take

$$\begin{aligned} P(x) &= \Phi_2(x)\Phi_{36}(x) = (x+1)(x^{12} - x^6 + 1), \\ Q(x) &= \Phi_1(x)^3\Phi_3(x)\Phi_6(x)\Phi_{18}(x). \end{aligned}$$

Both polynomials are monic integral of degree 13, have disjoint spectra, and have constant terms $+1$ and -1 . Let A, B be their companion matrices in the paper’s convention. They preserve a rational form of signature $(12, 1)$ and generate a primitive irreducible group, as checked on the next page. Levelt’s realization is used in the same way as in the paper.

Why the pair is absent from the list

In the $\mathcal{M}$ families, the underlying factors $x^r \pm 1$ have $r \leq n$, with roots ± 1 added or removed. The $\mathcal{N}$ families use quotients of factors $x^r - 1$; their largest numerator exponent is

$$\ell(k+1) = m + \ell \leq 2m \leq 2n, \quad \ell k = m \in \{n, n-1, n-2\}.$$

Consequently, at $n = 13$ every root in the list has order at most 26. Our P contains primitive *36th* roots. This excludes the pair even after exchanging P, Q . A half-shift preserves order 36 for these roots. Other scalar shifts cannot give integral characteristic polynomials: $\operatorname{tr} A = -1$ and $\operatorname{tr} B = 3$, so a unit scalar giving real traces must be ± 1 . An exact comparison of all 19 possible listed P -shapes gives the same exclusion.

Source of the omission. Start with the interlacing seed $\Phi_{12}(x)$ and $(x^2 - 1)\Phi_6(x)$; replace x by x^3 , then apply the proof’s Case 2 modification at 0 and $1/2$. Substitutions of sporadic seeds can raise the dimension, so bounding the *seed* dimension does not exclude them above 9.

Verification of the hypotheses

1. Exact hyperbolic form. Set $C = A^{-1}B$, $u = (C - I)e_{13}$ and $O = [u, Bu, \dots, B^{12}u]$. Direct integer arithmetic gives

$$C^2 = I, \quad \text{rank}(C - I) = 1, \quad u_{13} = -2, \quad \det O = -4608.$$

Define the symmetric Toeplitz matrix K by the first row

$$(-2, -4, -8, -13, -20, -28, -36, -44, -52, -59, -64, -68, -72).$$

Then $F = O^{-t}KO^{-1}$ satisfies $A^tFA = B^tFB = F$ and $72F$ is integral. Its signature can be checked without numerical eigenvalues: the leading principal determinants of K are

$$\begin{aligned} &-2, -12, -72, -324, -1458, -3645, -8748, \\ &-16848, -31680, -34752, -38080, -41408, -4608. \end{aligned}$$

They are all negative and nonzero. The rational LDL^t pivots are the consecutive ratios (with initial determinant 1), hence there is precisely one negative pivot and twelve positive ones.

2. Irreducibility. The image of $C - I$ is $\langle u \rangle$ and its kernel is $\ker(e_{13}^t)$. If an A, B -invariant complex subspace W is not fixed pointwise by C , it contains u and thus its full B -orbit basis. Otherwise $W \subset \ker(e_{13}^t)$. The matrix with rows $e_{13}^t B^j$, $0 \leq j \leq 12$, has determinant 1; B -invariance then forces $W = 0$.

3. Primitivity. In an irreducible group any permuted direct-sum blocks have equal dimension. Since 13 is prime, nontrivial blocks would be lines, making B monomial. Every invertible complex monomial matrix is semisimple: each weighted cycle has polynomial $x^r - a$, $a \neq 0$, with simple roots. But B is companion, so its minimal polynomial is Q , which contains $(x - 1)^3$. This contradiction excludes imprimitivity.

Consequences and remaining checks

The parameter list needs enlargement. This note makes no assertion that the resulting abstract group cannot also arise from other generators in a listed family. It makes no arithmeticity or thinness claim for the new pair.

The central component-image argument in Proposition 4.4 survives the source check. Lemma 4.6 additionally needs an infinite reflection quotient; its transfer to H uses the preceding finite-order hypothesis on the chosen B . Checking these conditions in Theorem 4.7 requires tracing the Vinberg/Nikulin inputs and reproducing the finite lattice exclusions. Generator swaps, signs and the appendix's displayed matrix identities can be checked directly by exact algebra. The appendix's congruence-subgroup and geometric finite-index certificates require more work. **An overall verdict on the thinness applications and appendix remains pending.**

All displayed matrix identities, determinants and family comparisons were checked with exact rational arithmetic. The analytic proof of Levelt's theorem was accepted as an external input. The audited edition and proof-input cutoff (received 11 August 2013) remain fixed; later repairs have not been surveyed.