

Fuchs–Meiri–Sarnak headline families: an exceptional lattice

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Finding. In Theorem 1(ii), the dimension-5 orbit lattice is *exactly* one of Nikulin’s reflective exceptions. The claimed exclusion of every small-dimensional lattice in Theorem 4.7(4) is therefore false. This is a failure of the thinness certificate, **not a proof that this hypergeometric group is arithmetic**.

The actual member. Its parameters and characteristic polynomials are

$$\alpha = (\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{1}{2}, \tfrac{5}{8}, \tfrac{7}{8}), \quad \beta = (0, 0, 0, \tfrac{1}{3}, \tfrac{2}{3}),$$

$$P(x) = (x+1)(x^4+1), \quad Q(x) = (x-1)^2(x^3-1).$$

Let A, B be the companion matrices, $C = A^{-1}B$, and $u = (C - I)e_5$. In the oriented orbit basis $u, (-A)u, \dots, (-A)^4u$, the normalized form has symmetric Toeplitz Gram matrix K with first row

$$(-2, -3, -4, -5, -6).$$

This spans the same integral lattice as the original B -orbit basis; the transition is unimodular. The lattice has signature $(4, 1)$, determinant -64 and nonunit Smith factors $(2, 2, 16)$.

An integral isometry, not just matching invariants. Put

$$M = \begin{pmatrix} -5 & -1 & -1 & 1 & 1 \\ 8 & 1 & 2 & -2 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & -1 & -1 \end{pmatrix}.$$

Then $\det M = -1$ and

$$M^t K M = \langle -16 \rangle \oplus \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix} = \langle -16 \rangle \oplus D_4.$$

Nikulin’s original Theorem 4 includes this lattice, with reversed signs. Hence $O(L)/R_2(L)$ is *finite*. Graph connectivity gives finite image in that quotient, and cannot supply the required infinite-index conclusion here. The D_4 discriminant group is $(\mathbb{Z}/2)^2$, so the exceptional factor entry $4 \cdot 16$ in FMS Table 1 does not describe its correct group $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/16$.

What survives. Making the generator choices explicit and checking the cited classification gives the intended thinness argument for family (i) in every odd $n \geq 5$ and family (ii) in every odd $n \geq 7$. The exceptional $n = 5$ group is primitive and hyperbolic, so the standing hypotheses do not remove it. A different argument is needed to settle this member. This issue is separate from the omitted parameter family in our earlier note.

Sources and checks. FMS, JEMS **16** (2014), Theorem 1 p.1619, Table 1 p.1635, Theorem 4.7 pp.1643–1645: [publisher PDF](#). Nikulin, Doklady **248**(6) (1979), Theorem 4 p.1308: [original article](#). All matrix, lattice-transition and Smith-factor checks used exact arithmetic. Uniform arguments are recorded separately; the finite computations alone are not used as an infinite-family proof.