NOTE FOR THE AUTHORS

Comments on Cubulating random quotients of hyperbolic cubulated groups

For David Futer and Daniel T. Wise

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These notes concern **arXiv:2106.04497v3** [1]. All page numbers and theorem numbers refer to that version. The journal article has not been compared line by line, so some of the points below may already have been addressed there.

The principal question is about the normalization of a self-overlap in the proof of Lemma 4.11(2). An explicit example shows why a choice of representative modulo the cyclic stabilizer is needed before the counting argument. A second question concerns the cocompactness conclusion in the full stated generality of Theorem 3.5. The remaining points are local corrections or requests to make conventions explicit.

None of the examples below is a counterexample to either main random-quotient theorem. The density calculations leading to 41 and $40\lambda + 1$ are consistent with the intended strategy; the main request is to clarify and complete the self-overlap count on which they depend.

1 Normalization in the self-overlap count

Manuscript location: Definition 4.8, p. 16; Lemma 4.11(2), p. 20; Lemma 4.13, pp. 21–23.

Let g stabilize an axis L, and write

$$q : L \longrightarrow w = \langle g \rangle \backslash L$$

for the quotient onto the relator's parameter circle. For a loose self-piece $s \subset L$ and its companion $s' \subset hL$, Definition 4.8 sets

$$u = s \cap h^{-1}s'.$$

In the proof of Lemma 4.11(2), the remaining portion $v' = s' \setminus h(u)$, with orientation adjusted if necessary, is then treated as disjoint from s on the relator circle. This is used to choose a fundamental domain of the form $y_1 v' y_2 s$.

The point requiring clarification is that hL is unchanged under $h \mapsto hg^N$, whereas

$$(hg^N)^{-1}s' = g^{-N}h^{-1}s'.$$

Thus the lifted overlap can be made empty without changing the two occurrences on w. More precisely, the occurrence represented by s' is $q(h^{-1}s')$, and this also remains unchanged. Disjoint intervals in L need not project to disjoint intervals on w.

No normalization of h modulo $\langle g \rangle$ is apparent in the preceding setup. The choice placing the initial endpoint of s near a fixed basepoint does not impose one: both the endpoint and the pair of axes remain fixed under this change of h.

An example in the range used later

In the unit-edge Cayley tree of $F(a,b)$, let

$$g = a^m b^{3m}, \qquad m \geq 4,$$

and let L be its axis through 1. Then

$$L \cap aL = [a, a^m].$$

For $J = 1$, the maximal loose piece on L and its companion on aL are

$$s = [1, a^m b], \quad |s| = m+1, \qquad s' = [a, a^m], \quad |s'| = m-1.$$

Each endpoint of s is at distance one from the corresponding endpoint of s'; extending s by another edge at either end gives distance two from aL.

Set $h_N = ag^N$, with $N \geq 2$. Then

$$h_N L = aL, \qquad h_N^{-1} s' = g^{-N}[1, a^{m-1}], \qquad s \cap h_N^{-1} s' = \varnothing.$$

Nevertheless,

$$q(h_N^{-1} s') = q([1, a^{m-1}]) \subset q(s).$$

Consequently $u = \varnothing$ and $v' = s'$, but the two relevant occurrences overlap on the parameter circle. The asserted disjointness does not follow for this representative h_N.

Here g is not a proper power: its cyclic word has just one transition from an a-run to a b-run, which a nontrivial cyclic power would repeat. Moreover,

$$\ell = |w| = 4m, \qquad d = m \leq \ell/3, \qquad d > 2J, \qquad |s| = m+1 < |w|/2.$$

The example therefore lies in the quantitative range used downstream and does not rely on proper powers or on a piece longer than half the relator. To satisfy the literal subdivision convention, subdivide the rose and double all displayed lengths and J.

Requested clarification and a possible reorganization

Would defining overlap on the parameter circle, or choosing h to realize that overlap, make the intended argument precise? The normalization needs to occur *before* the division into small- and large-overlap cases: changing h can change whether $|u| \leq R$.

For the range $d \leq \ell/3$ needed later, one possible approach is to discard relators of length at most $2d + C$, then truncate a long piece and its companion to suitable lengths near d. Here C depends only on the fixed geometric constants. Using the conjugacy-class estimates already in the paper, the discarded classes contribute at most

$$O\Big(\ell e^{-b(\ell-2d-C)}\Big) = O\Big(\ell e^{-bd}\Big).$$

If the two retained arc lengths sum to less than $|w|$, their circular intersection is empty or connected and can be represented by a single lifted overlap. One can then choose h modulo $\langle g \rangle$ before applying the two counting cases.

This is a possible repair route, not a completed replacement proof. It would be helpful to spell out the truncation, the normalized case division, and the resulting count. A bound $C_0 \ell^r e^{-bd}$ with any fixed r would suffice for the subsequent density argument: for every $\varphi < 1$, the polynomial factor is absorbed into $e^{b(1-\varphi)d}$ once d exceeds a sufficiently large multiple of $\log \ell$.

2 Cocompactness and the scope of Theorem 3.5

Manuscript location: Theorems 3.5 and 3.8, pp. 7–12; final paragraph of the proof on p. 12; Figure 1's caption.

At the end of the proof of Theorem 3.5, cocompactness of the dual action is attributed to compactness of X. Could the argument or reference establishing this conclusion in the stated generality be supplied?

Theorem 3.8, corresponding to Theorem 5.44 and Corollary 5.45 of [2], gives torsion stabilizers. **Metric properness does have an available bridge:** Remark 5.46 of [2] applies because the finitely many compact relators have uniformly bounded diameter for each fixed presentation. This does not require a diameter bound uniform in the random length parameter ℓ. Properness alone, however, does not establish cocompactness of the dual action.

In the hyperbolic setting of the principal applications, the following seems to supply the needed argument. Lemma 3.4 makes the quotient hyperbolic. Corollary 5.30 of [2] gives quasi-isometrically embedded wall carriers; their cocompact stabilizers are therefore quasiconvex. There are finitely many wall orbits, so the cocompactness criterion recorded in Proposition 7.4 of [2] applies.

The standalone Theorem 3.5 does not assume that $\pi_1 X$ is hyperbolic, whereas the caption to Figure 1 describes its nonhyperbolic example as differing from the setting of that theorem. Is a hyperbolicity hypothesis intended, or is there a more general cocompactness argument to cite? This is a question about the scope and justification of the deterministic statement, rather than a counterexample to it.

Two smaller clarifications in the same proof would be useful:

- In checking Theorem 3.8(4), choose u_2 explicitly dual to an edge traversed by κ near its midpoint, and explain why the resulting wall U_2 separates the endpoints. The existing bounds exclude the antipodal hyperplane from κ; geodesicity and separation of distinct lifts exclude a second crossing of u_2. The mesh-sized adjustment is absorbed by the strict $\alpha < 1/20$ margin.
- In the final sentence of the proof of Theorem 3.5, the space on which the resulting action is free should be the dual CAT(0) cube complex, rather than $\widetilde{X^*}$.

3 The wall-piece estimate when the growth exponent is zero

Manuscript location: Proposition 4.7, pp. 15–16; equation (4.2) and the final choice of C.

The calculation initially has denominator

$$1 - \exp\big(-[a\alpha + b(1-\alpha)]\big),$$

but the final constant is set to $B/[A(1-e^{-a})]$. This is undefined when $a = 0$, including the cyclic-hyperplane examples. Retaining the earlier denominator gives the finite constant

$$C_\alpha = \frac{B}{A\left(1 - e^{-[a\alpha + b(1-\alpha)]}\right)}.$$

For the main application $\alpha < 1/20$, its denominator is bounded below by $1 - e^{-19b/20}$. Thus a uniform bound over the needed interval is available; otherwise the proposition's constant can be allowed to depend on the fixed α.

There is a related qualification in using pure exponential growth for the hyperplane stabilizer. An infinite cyclic group has exponent zero but linearly growing balls. The count here is at a specified

length, so bounded-width annuli, which contain uniformly bounded numbers of orbit points in the cyclic case, can be used instead. Alternatively, an $a + \varepsilon$ estimate is sufficient for the final theorem, choosing $\varepsilon > 0$ within the strict margin in $c < \alpha(b - a)$. These corrections need not change the final density threshold.

4 Conventions and preliminary hypotheses

Equality of lifted relators (Definition 3.2, p. 7)

Could the exclusion of equal lifts specify that relator identity is retained? Under the standard indexed-elevation convention, coincident images of elevations of different relators still contribute a cone-piece, unless those relators have already been explicitly identified. If equality means only equality of image subsets of $\widetilde{X}$, different finite covers of the same closed loop would have their common lifted axis excluded. The standard convention, made explicit for example in [3, Definition 3.5 and Remark 3.6], appears to be intended; stating it would remove the ambiguity.

The metric hypotheses for growth (pp. 5–6)

The assertion accompanying Definition 2.2 that a proper cocompact action on an arbitrary metric space gives at-most-exponential orbit growth needs a geometric hypothesis. For example, equip F_2 with the left-invariant metric

$$d(g, h) = \sqrt{|g^{-1}h|},$$

where $|\cdot|$ is word length. The action is proper and transitive, but its radius-n balls have size comparable to 3^{n^2}. The cited growth-gap results use a graph in [4, Theorem 1.1] and a proper geodesic hyperbolic space in [5]. Could Definition 2.2 and Theorem 2.3 state the corresponding hypotheses? The geometric spaces used in the main applications have the needed additional structure.

Passing to an essential core (pp. 25 and 31)

The assertions that both a and b remain unchanged under the essential-core reduction seem stronger than needed. For the usual core, the safe relation is

$$b_{\mathrm{core}} = b, \qquad a_{\mathrm{core}} \le a.$$

The paper's definition of an essential hyperplane is weaker than the two-deep-halfspaces condition used in the standard construction [6].

For an illustration, let X_0 be a rank-three rose and Y its rank-two subrose. Attach $Y \times [0, 1]$ along $Y \times \{0\}$. The resulting compact nonpositively curved complex has fundamental group F_3. The collar's mid-hyperplane has stabilizer F_2, with exponent $\log 3$, and is essential in the paper's sense: the whole universal cover is not contained in a bounded neighborhood of one lift. Pruning the collar leaves X_0, whose hyperplane stabilizers are trivial. The maximal exponent drops from $\log 3$ to zero, while $b = \log 5$ is unchanged. A further product with a compact interval supplies an inessential hyperplane as well, so the conditional reduction on p. 25 is triggered.

Keeping the original a as an upper bound suffices: decreasing a only improves the allowed density. An inequality in place of equality would avoid this issue without weakening the main theorem.

5 Local corrections

1. **Lemma 4.13, p. 23.** Combining the five bad-event estimates requires the sum of their constants, or five times their maximum, rather than their maximum alone.

2. **Lemma 4.13, Case 2.** Replacing J by $J + 2\delta$ also calls for enlarging the auxiliary overlap threshold; a common maximum of the required constants suffices.

3. **Lemma 4.15, pp. 23–24.** Proposition 4.14 is stated only for $d \le \ell/3$. It suffices to apply it at $d = 2q\ell$, which is in range because $q \le 1/6$, rather than at every $d \ge 2q\ell$.

4. **Proposition 5.7, p. 30.** The expression

$$-2(\eta + \lambda_2(K+1) + \varepsilon + \psi + 2\delta) - 2\eta$$

   simplifies to

$$-2(2\eta + \lambda_2(K+1) + \varepsilon + \psi + 2\delta).$$

   The printed simplification has δ in place of the final 2δ. The correction propagates through the subsequent additive constants and changes only the threshold M_2.

5. **Discussion after Corollary 1.2, p. 3.** The surface-group density should be strictly below $1/41$; the displayed theorem does not establish the endpoint. The rounded denominator 63.51 in Corollary 1.4, by contrast, lies safely inside the strict bound.

6 Checks on the stated quantitative bounds

The preceding questions do not reveal a numerical obstruction to the principal density bounds. In the antipodal-wall construction, the limiting inequality is $1/4 > 5\alpha$, so the strict $1/20$ hypothesis supplies the required margin.

For the probability argument, balancing the overlap bound $2q\ell$ against the minimum relator length $(1-q)\ell$ gives

$$2q = \alpha(1-q), \qquad q = \frac{\alpha}{\alpha+2}.$$

The pair union bound and the short-relator estimate both require $c < bq$. Letting α approach $1/20$ from below gives $b/41$; the wall-piece estimate separately gives $(b-a)/20$. Replacing the effective small-cancellation parameter by α/λ yields the denominators 20λ and $40\lambda + 1$ in Theorem 1.3. These calculations remain conditional on the self-overlap estimate discussed in Section 1.

In the pentagonal example, the displayed formula gives

$$\lambda \approx 1.5627191146, \qquad 40\lambda + 1 \approx 63.5087645853 < 63.51.$$

The rounding in Corollary 1.4 is therefore in the correct direction.

References

[1] D. Futer and D. T. Wise, *Cubulating random quotients of hyperbolic cubulated groups*, arXiv:2106.04497v3 (2023). Published in Trans. Amer. Math. Soc. Ser. B **11** (2024), 622–666. These notes use the arXiv pagination. https://arxiv.org/abs/2106.04497v3.

[2] D. T. Wise, *The Structure of Groups with a Quasiconvex Hierarchy*, Annals of Mathematics Studies 209, Princeton University Press, 2021. https://doi.org/10.1515/9780691213507.

[3] M. Arenas, *Asphericity of cubical presentations: the 2-dimensional case*, arXiv:2305.06296v3, Definition 3.5 and Remark 3.6. In the published version, IMRN 2024, no. 7, 5524–5547, the corresponding numbers are Definition 2.5 and Remark 2.6. https://arxiv.org/abs/2305.06296v3.

[4] F. Dahmani, D. Futer, and D. T. Wise, *Growth of quasiconvex subgroups*, Math. Proc. Cambridge Philos. Soc. **167** (2019), 505–530. https://arxiv.org/abs/1602.08085.

[5] K. Matsuzaki, Y. Yabuki, and J. Jaerisch, *Normalizer, divergence type, and Patterson measure for discrete groups of the Gromov hyperbolic space*, Groups Geom. Dyn. **14** (2020), 369–411. https://arxiv.org/abs/1511.02664.

[6] P.-E. Caprace and M. Sageev, *Rank rigidity for CAT(0) cube complexes*, Geom. Funct. Anal. **21** (2011), 851–891, Section 3 and Proposition 3.5. https://arxiv.org/abs/1005.5687.