

Effective version of Ratner’s equidistribution theorem for $\mathrm{SL}(3, \mathbb{R})$

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Abstract

In this paper, we will prove an effective version of Ratner’s equidistribution theorem for unipotent orbits in $\mathrm{SL}(3, \mathbb{R})/\mathrm{SL}(3, \mathbb{Z})$ with a natural Diophantine condition.

1. Introduction

1.1. *Ratner’s theorem and its effective versions.* In 1991, Ratner [Rat91b] proved the following fundamental theorem on equidistribution of unipotent orbits in homogeneous spaces:

THEOREM 1.1. *Let G be a Lie group and Γ be a lattice in G , namely, the homogeneous space $X = G/\Gamma$ admits a G -invariant probability measure. Let $U = \{u(r) : r \in \mathbb{R}\}$ be a one-parameter unipotent subgroup of G . For any $x \in X$, the closure of the U -orbit Ux of x is a closed L -orbit Lx for some Lie subgroup $L \subset G$. Moreover, Ux is equidistributed in Lx with respect to the unique L -invariant probability measure μ_L , namely, for any $f \in C_c^\infty(X)$ (where $C_c^\infty(X)$ denotes the space of smooth functions on X with compact support),*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(u(r)x) dr = \int_{Lx} f d\mu_L.$$

Since Ratner’s proof relies on her measure classification theorem ([Rat90a], [Rat90b], [Rat91a]) which proves a conjecture by Dani [Dan81], it does not tell how fast the orbit becomes equidistributed. Therefore, a natural question is whether we can make the equidistribution effective, namely, giving an explicit

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upper bound on the difference

$$\left| \frac{1}{T} \int_{-T/2}^{T/2} f(u(r)x) dr - \int_{Lx} f d\mu_L \right|$$

for given $T > 0$. Moreover, effective versions of equidistribution of U -orbits have many applications to number theory; see [LM14], [EM22], [LM23], [LMW23], [Ven10], [NV21], [EMV09], [CY24], [BV16] and references therein for details. As a result, proving effective versions of Ratner's theorem has attracted much attention and has been a major challenge in homogeneous dynamics.

In recent years, people have made significant progress in establishing effective equidistribution results. For G being nilpotent, the effective version of Ratner's theorem was established by Green and Tao [GT12]. For $G = \mathrm{SL}(2, \mathbb{R})$, since the one-parameter unipotent subgroup is horospherical with respect to the diagonal subgroup, one can apply the thickening argument developed in Margulis's thesis (cf. [Mar04], [KM12]) and spectral gap results for unitary representations of semisimple groups to establish effective equidistribution. See [Sar81], [Bur90], [FF03], [SU15], [Str04] and [Str13] for works in this setting. In fact, equidistribution for this case was proved before Ratner's theorem; see [Fur73], [DS84]. Using similar arguments, one can also establish effective equidistribution results for horospherical unipotent orbits in homogeneous spaces. See [KM12], [Edw21] for results in this setting. Using techniques from analytic number theory, Strömbergsson [Str15] established the effective equidistribution for $G/\Gamma = \mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2 / \mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ and U being a one-parameter unipotent (and horospherical) subgroup in the semisimple part. Building on Strömbergsson's result, Chow and the author [CY24] proved an effective equidistribution result for a special family of one-parameter unipotent orbits in $G/\Gamma = \mathrm{SL}(3, \mathbb{R}) / \mathrm{SL}(3, \mathbb{Z})$. As an application, we proved that Gallagher's theorem in multiplicative Diophantine approximation holds for almost every point on any given planar straight line. The reader is also referred to [BV16] for a similar effective equidistribution result that has applications to number theory. Strömbergsson's result was recently generalized by Kim [Kim24] to $\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n / \mathrm{SL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ with the unipotent subgroup being horospherical in the semisimple part. Recently, Lindenstrauss, Mohammadi and Wang [LMW23] established effective Ratner's theorem for one-parameter unipotent orbits in G/Γ where $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$. The reader is referred to [EMV09] and [EMMV20] for effective equidistribution of closed orbits of maximal semisimple subgroups which can be regarded as an effective version of a result by Mozes and Shah [MS95].

Concerning effective density, there are also significant results. Lindenstrauss and Margulis proved [LM14] an effective density result for one-parameter unipotent orbits in $\mathrm{SL}(3, \mathbb{R}) / \mathrm{SL}(3, \mathbb{Z})$ and applied it to prove an effective

version of Oppenheim's conjecture. Recently, Lindenstrauss and Mohammadi [LM23] established an optimal effective density result for unipotent orbits in G/Γ for $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SL}(2, \mathbb{C})$.

1.2. *Notation.* Throughout this paper, we will fix the following notation:

- (1) For a normed vector space V and $r > 0$, let $B_V(r)$ denote the set of vectors in V with norm $\leq r$.
- (2) For a Lie group J and $r > 0$, let $B^J(r)$ denote the r -neighborhood of the identity in J .
- (3) Given a quantity $\mathcal{A}$, let $O(\mathcal{A})$ denote a quantity whose absolute value is $\leq C\mathcal{A}$, where $C > 0$ is an absolute constant. By $\mathcal{A} \ll \mathcal{B}$, we mean $\mathcal{A} = O(\mathcal{B})$. By $\mathcal{A} \asymp \mathcal{B}$, we mean $\mathcal{A} \ll \mathcal{B}$ and $\mathcal{B} \ll \mathcal{A}$.
- (4) For a normed vector space V and a quantity $\mathcal{A}$, let $O_V(\mathcal{A})$ denote a vector in V whose norm is $O(\mathcal{A})$.
- (5) For a measurable subset $I \subset \mathbb{R}$, let $\mathrm{Leb}(I)$ denote the Lebesgue measure of I .
- (6) For a finite set S , let $|S|$ denote the cardinality of S .
- (7) For $L > 0$, let $[L]$ denote the interval $[-L/2, L/2]$.

1.3. *Main results.* In this paper we will focus on the following case: Let $G = \mathrm{SL}(3, \mathbb{R})$, $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, $X = G/\Gamma$. Let $U = \{u(r) : r \in \mathbb{R}\}$ be a one-parameter unipotent subgroup of G defined as follows:

$$(1.1) \quad u(r) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix}.$$

Let μ_G denote the unique G -invariant probability measure on X .

For $\varpi > 0$, let us define

$$(1.2) \quad X_\varpi := \{x = g\Gamma \in X : \|g\mathbf{v}\| \geq \varpi, \forall \mathbf{v} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\} \text{ or } \wedge^2 \mathbb{Z}^3 \setminus \{\mathbf{0}\}\}.$$

By Mahler's criterion, every compact subset of X is contained in some X_ϖ . For $t \in \mathbb{R}$, let us denote

$$(1.3) \quad a(t) := \begin{bmatrix} 1 & & \\ & e^{t/2} & \\ & & e^{-t/2} \end{bmatrix} \in G,$$

and

$$(1.4) \quad a_0(t) := \begin{bmatrix} e^{t/3} & & \\ & e^{-t/6} & \\ & & e^{-t/6} \end{bmatrix} \in G.$$

We will prove the following result for effective equidistribution of U -orbit:

THEOREM 1.2. *There exist $\eta > 0$ and $t_0 > 1$ such that for any $t \geq t_0$ and any $x_0 \in X$, one of the following holds:*

(1) *For any $f \in C_c^\infty(X)$, we have*

$$\left| \int_{[1]} f(u(re^t)x_0) dr - \int f d\mu_G \right| \ll e^{-\eta t} \|f\|_S,$$

where $\|\cdot\|_S$ denotes a fixed Sobolev norm;

(2) *$a(-t)a_0(\ell)x_0 \notin X_{e^{-|\ell|/3}}$ for some $0.99t \leq |\ell| \leq 1.01t$.*

Proving [Theorem 1.2](#) is equivalent to proving the following theorem:

THEOREM 1.3. *Let $\eta, t_0 > 0$ be as above. For any $t \geq t_0$, and any $x_0 \in G/\Gamma$, one of the following holds:*

(1) *For any $f \in C_c^\infty(X)$, we have*

$$\left| \int_{[1]} f(a(t)u(r)x_0) dr - \int f d\mu_G \right| \ll e^{-\eta t} \|f\|_S;$$

(2) *$a_0(\ell)x_0 \notin X_{e^{-|\ell|/3}}$ for some $0.99t \leq |\ell| \leq 1.01t$.*

In fact, it follows from the equality

$$\int_{[1]} f(a(t)u(r)x_0) dr = \int_{[1]} f(u(re^t)y_0) dr,$$

where $y_0 = a(t)x_0$.

Using [Theorem 1.3](#), we can prove the following two corollaries on effective equidistribution of expanding curves under translates of diagonal subgroups:

COROLLARY 1.4. *Let us denote*

$$a_1(t) := a(t)a_0(t) = \begin{bmatrix} e^{t/3} & & \\ & e^{t/3} & \\ & & e^{-2t/3} \end{bmatrix}.$$

Let $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ be a vector with Diophantine exponent $\omega(\mathbf{w}) = \kappa \leq 0.6$; namely, for any positive integer $q \in \mathbb{Z}_+$, we have

$$\max\{\langle qw_1 \rangle, \langle qw_2 \rangle\} \gg q^{-\kappa},$$

where $\langle \cdot \rangle$ denotes the distance to the nearest integer. For $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, let us denote

$$n(\mathbf{v}) := \begin{bmatrix} 1 & & v_2 \\ & 1 & v_1 \\ & & 1 \end{bmatrix}.$$

Let us define

$$\varphi : [1] \rightarrow \mathbb{R}^2$$

by $\varphi(r) := (w_1 r + w_2, r)$. Then there exist $\eta, t_0 > 0$, such that for any $t \geq t_0$ and any $f \in C_c^\infty(X)$, we have

$$\left| \int_{[1]} f(a_1(t)n(\varphi(r))\Gamma) dr - \int f d\mu_G \right| \ll e^{-\eta t} \|f\|_S.$$

Proof assuming Theorem 1.3. Note that

$$a_1(t)n(\varphi(r)) = z(w_1)a(t)u(r)a_0(t)n^*(-w_1, w_2),$$

where

$$z(w_1) := \begin{bmatrix} 1 & w_1 & \\ & 1 & \\ & & 1 \end{bmatrix},$$

and

$$n^*(-w_1, w_2) := \begin{bmatrix} 1 & -w_1 & w_2 \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Then for any $f \in C_c^\infty(X)$ with zero integral, we have that

$$\int_{[1]} f(a_1(t)n(\varphi(r))\Gamma) dr = \int_{[1]} f_{w_1}(a(t)u(r)a_0(t)n^*(-w_1, w_2)\Gamma) dr,$$

where $f_{w_1}(x) = f(z(w_1)x)$. Note that since μ_G is G -invariant, f_{w_1} also has zero integral.

Let $x_0 = a_0(t)n^*(-w_1, w_2)\Gamma$. Note that the Diophantine condition on $\mathbf{w}$ ensures that $a_0(\ell)x_0 \in X_{e^{-|\ell|/3}}$ for any $0.99t \leq |\ell| \leq 1.01t$. Applying Theorem 1.3 with $x_0 = a_0(t)n^*(-w_1, w_2)\Gamma$ and f_{w_1} , we have that

$$\left| \int_{[1]} f_{w_1}(a(t)u(r)a_0(t)n^*(-w_1, w_2)\Gamma) dr \right| \ll e^{-\eta t} \|f_{w_1}\|_S.$$

Noting that $\|f_{w_1}\|_S \ll \|f\|_S$, we complete the proof. $\square$

Remark 1.5. The qualitative version of Corollary 1.4 for $\omega(\mathbf{w}) \leq 2$ was proved in [KdSSY24] using Ratner's theorem. The authors also proved that the equidistribution does not hold for $\omega(\mathbf{w}) > 2$.

COROLLARY 1.6. *Let us use the same notation as in Corollary 1.4. Let $\psi : [1] \rightarrow \mathbb{R}^2$ be a smooth non-degenerate curve in $\mathbb{R}^2$; namely, derivatives of ψ span the whole space $\mathbb{R}^2$ at every $r \in [1]$. Then there exist $\eta, t_0 > 0$ such that for any $t \geq t_0$ and any $f \in C_c^\infty(X)$ we have*

$$\left| \int_{[1]} f(a_1(t)n(\psi(r))\Gamma) dr - \int f d\mu_G \right| \ll e^{-\eta t} \|f\|_S.$$

Proof assuming Theorem 1.3. Let us write $\psi(r) = (r, \psi_2(r))$. Let us divide $[0, 1]$ into small pieces of size $e^{-t/2}$. Let us fix a small piece $\Delta(r_0) = [r_0 - 1/2e^{-t/2}, r_0 + 1/2e^{-t/2}]$. Then

$$\{a_1(t)n(\psi(r))\Gamma : r \in \Delta(r_0)\}$$

can be approximated by

$$\{z(\psi'_2(r_0))a(t/2)u(r)x_0 : r \in [-1/2, 1/2]\}$$

with exponentially small error, where

$$z(\psi'_2(r_0)) = \begin{bmatrix} 1 & \psi'_2(r_0) & \\ & 1 & \\ & & 1 \end{bmatrix},$$

as defined in Corollary 1.4, and

$$x_0 = a_0(t/2)a_1(t/2)z(-\psi'_2(r_0))n(\psi(r_0))\Gamma.$$

Then if we can show that

$$\{a(t/2)u(r)x_0 : r \in [1]\}$$

effectively equidistributed, we will get that

$$\{a_1(t)n(\psi(r)) : r \in \Delta(r_0)\}$$

is effectively equidistributed. By Theorem 1.3, if we have that for any $0.49t \leq |\ell| \leq 0.51t$,

$$a_0(\ell)a_0(t/2)a_1(t/2)z(-\psi'_2(r_0))n(\psi(r_0))\Gamma \in X_{e^{-t/10}},$$

then we are done. Now for a fixed $|\ell| \in [0.49t, 0.51t]$, let us estimate the measure of

$$\mathbf{m}_\ell := \{r_0 \in [1] : a_0(\ell)a_0(t/2)a_1(t/2)z(-\psi'_2(r_0))n(\psi(r_0))\Gamma \notin X_{e^{-t/10}}\}.$$

By [BKM01, Theorem 1.4], we have that

$$\text{Leb}(\mathbf{m}_\ell) = O(e^{-\eta_2 t})$$

for some constant $\eta_2 > 0$ independent of ℓ . Let us remove all $\mathbf{m}_\ell$ from $[1]$ and get a subset $\mathbf{m} \subset [1]$. Then we have that

$$\text{Leb}([1] \setminus \mathbf{m}) = O(te^{-\eta_2 t}),$$

and for any $r_0 \in \mathbf{m}$,

$$\{a_1(t)n(\psi(r))\Gamma : r \in \Delta(r_0)\}$$

is effectively equidistributed in X . Combining these two facts we conclude that the whole orbit

$$\{a_1(t)n(\psi(r))\Gamma : r \in [1]\}$$

is effectively equidistributed. This completes the proof. $\square$

Remark 1.7. The ineffective version of [Corollary 1.6](#) was proved in [\[Sha09\]](#) using Ratner's theorem.

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2. Preliminaries

In this section we recall some basic facts on $\mathrm{SL}(3, \mathbb{R})$ and its Lie algebra, which will be used in the proof of our main theorem.

Let $G = \mathrm{SL}(3, \mathbb{R})$, and $H \subset G$ be the following subgroup of G :

$$(2.1) \quad H := \left\{ \begin{bmatrix} 1 & & \\ & h & \\ & & 1 \end{bmatrix} \in G : h \in \mathrm{SL}(2, \mathbb{R}) \right\}.$$

Clearly H is isomorphic to $\mathrm{SL}(2, \mathbb{R})$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H , respectively.

Consider the adjoint action of H on $\mathfrak{g}$; we have the following decomposition:

$$(2.2) \quad \mathfrak{g} = \mathfrak{h} + \mathfrak{r}_0 + \mathfrak{r}_1 + \mathfrak{r}_2,$$

where $\mathfrak{r}_0, \mathfrak{r}_1, \mathfrak{r}_2$ are irreducible subrepresentations with respect to the adjoint action of H , and $\dim \mathfrak{r}_1 = \dim \mathfrak{r}_2 = 2$, $\dim \mathfrak{r}_0 = 1$. In particular, $\mathfrak{r}_0 = \mathbb{R}\mathfrak{a}_0$, where

$$\mathfrak{a}_0 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & -1/6 \end{bmatrix};$$

$\mathfrak{r}_1 = \mathbb{R}\mathbf{v}_1 + \mathbb{R}\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$\mathfrak{r}_2 = \mathbb{R}\mathbf{w}_1 + \mathbb{R}\mathbf{w}_2$. where

$$\mathbf{w}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The adjoint actions of H on $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are given as follows: the adjoint action of H on $\mathfrak{r}_1$ is the same as the standard action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ if we choose $\{\mathbf{v}_1, \mathbf{v}_2\}$ as the basis; for h corresponding to $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathbf{w} = x_1\mathbf{w}_1 + x_2\mathbf{w}_2$,

$$\mathrm{Ad}(h)\mathbf{w} = (ax_1 - bx_2)\mathbf{w}_1 + (-cx_1 + dx_2)\mathbf{w}_2.$$

Let $\{\mathfrak{a}, \mathbf{u}, \mathbf{u}^*\}$ denote the standard basis of $\mathfrak{h}$, where

$$\mathfrak{a} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \quad \mathbf{u} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}^* := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let us denote

$$\begin{aligned} A &:= \{a(t) = \exp(t\mathfrak{a}) : t \in \mathbb{R}\} \subset H, \\ U &:= \{u(r) := \exp(r\mathbf{u}) : r \in \mathbb{R}\} \subset H, \\ U^* &:= \{u^*(r) := \exp(r\mathbf{u}^*) : r \in \mathbb{R}\} \subset H, \end{aligned}$$

and

$$A_0 := \{a_0(t) := \exp(t\mathfrak{a}_0) : t \in \mathbb{R}\} \subset Z_G(H).$$

Note that the definitions of $a(t)$ and $a_0(t)$ here match (1.3) and (1.4), respectively. The adjoint action of $a_0(t)$ on $\mathfrak{g}$ is as follows:

$$\begin{aligned} \mathrm{Ad}(a_0(t))\mathbf{h} &= \mathbf{h}, \text{ for } \mathbf{h} \in \mathfrak{h}, \\ \mathrm{Ad}(a_0(t))\mathbf{v} &= e^{-t/2}\mathbf{v}, \text{ for } \mathbf{v} \in \mathfrak{r}_1, \end{aligned}$$

and

$$\mathrm{Ad}(a_0(t))\mathbf{w} = e^{t/2}\mathbf{w}, \text{ for } \mathbf{w} \in \mathfrak{r}_2.$$

Let us denote

$$(2.3) \quad b(t) := a(t)a_0(-t) = \begin{bmatrix} e^{-t/3} & & \\ & e^{2t/3} & \\ & & e^{-t/3} \end{bmatrix} \in G,$$

and

$$(2.4) \quad a_1(t) := a(t)a_0(t) = \begin{bmatrix} e^{t/3} & & \\ & e^{t/3} & \\ & & e^{-2t/3} \end{bmatrix} \in G.$$

Let us denote

$$\mathbf{r}^+ := \mathbb{R}\mathbf{w}_1 + \mathbb{R}\mathbf{v}_1, \quad \text{and} \quad \mathbf{r}^- := \mathbb{R}\mathbf{w}_2 + \mathbb{R}\mathbf{v}_2.$$

Let us denote by $p_{1,2} : \mathfrak{g} \rightarrow \mathfrak{r}_1 + \mathfrak{r}_2$ the projection of $\mathfrak{g}$ to $\mathfrak{r}_1 + \mathfrak{r}_2$. For $j = 0, 1, 2$, let $p_j : \mathfrak{g} \rightarrow \mathfrak{r}_j$ denote the projection from $\mathfrak{g}$ to $\mathfrak{r}_j$. Let p_+ , p_- , $p_{\mathbf{w}_j}$, $p_{\mathbf{v}_j}$ and $p_{\mathbf{u}^*}$ denote the projection to $\mathbf{r}^+$, $\mathbf{r}^-$, $\mathbb{R}\mathbf{w}_j$, $\mathbb{R}\mathbf{v}_j$ and $\mathbb{R}\mathbf{u}^*$, respectively.

We will need the following simple lemma on the Lie algebra of $\mathrm{SL}(3, \mathbb{R})$:

LEMMA 2.1. *For $\mathbf{w}^+ \in \mathbf{r}^+$ and $\mathbf{w}^- \in \mathbf{r}^-$,*

$$\begin{aligned} [\mathbf{w}^+, \mathbf{w}^-] &\in \mathbb{R}\mathbf{a} + \mathbb{R}\mathbf{a}_0, \\ [\mathbf{w}^+, \mathbf{a}], [\mathbf{w}^+, \mathbf{a}_0] &\in \mathbf{r}^+, \\ [\mathbf{w}^-, \mathbf{a}], [\mathbf{w}^-, \mathbf{a}_0] &\in \mathbf{r}^-, \\ [\mathbf{w}^-, \mathbf{u}] &\in \mathbf{r}^+, [\mathbf{w}^-, \mathbf{u}^*] = \mathbf{0}, \end{aligned}$$

and

$$[\mathbf{w}^+, \mathbf{u}^*] \in \mathbf{r}^-, [\mathbf{w}^+, \mathbf{u}] = \mathbf{0}.$$

For $\mathbf{w}_1^+, \mathbf{w}_2^+ \in \mathbf{r}^+$,

$$[\mathbf{w}_1^+, \mathbf{w}_2^+] \in \mathbb{R}\mathbf{u}.$$

For $\mathbf{w}_1^-, \mathbf{w}_2^- \in \mathbf{r}^-$,

$$[\mathbf{w}_1^-, \mathbf{w}_2^-] \in \mathbb{R}\mathbf{u}^*.$$

For $\mathbf{h} \in \mathfrak{h}$,

$$[\mathbf{a}_0, \mathbf{h}] = \mathbf{0}.$$

Also, we have

$$[\mathbf{a}, \mathbf{u}] = \mathbf{u}, \quad [\mathbf{a}, \mathbf{u}^*] = -\mathbf{u}^*, \quad \text{and} \quad [\mathbf{u}, \mathbf{u}^*] = 2\mathbf{a}.$$

Proof. The statements can be verified by direct calculation. $\square$

We also need the following lemmata concerning the Lie algebra coordinate of the product of two elements in G :

LEMMA 2.2. *Let $0 < \rho_1, \rho_2 < 10^{-10}$ be small constants. For $\mathbf{w} \in B_{\mathfrak{r}_1 + \mathfrak{r}_2}(\rho_1)$ and $\mathbf{h} \in B_{\mathbb{R}\mathbf{a}_0 + \mathfrak{h}}(\rho_2)$, we have that*

$$(2.5) \quad \exp(\mathbf{w}) \exp(\mathbf{h}) = \exp(\tilde{\mathbf{w}} + \tilde{\mathbf{h}}),$$

where $\tilde{\mathbf{w}} \in \mathfrak{r}_1 + \mathfrak{r}_2$ with $\|\mathbf{w} - \tilde{\mathbf{w}}\| \leq 2\rho_2\|\mathbf{w}\|$, and $\tilde{\mathbf{h}} \in \mathbb{R}\mathbf{a}_0 + \mathfrak{h}$ with

$$\|\mathbf{h} - \tilde{\mathbf{h}}\| \leq 2\rho_1\|\mathbf{h}\|.$$

Conversely, for $\tilde{\mathbf{w}} \in B_{\mathfrak{r}_1 + \mathfrak{r}_2}(\rho_1)$ and $\tilde{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathfrak{h}}(\rho_2)$, we have

$$(2.6) \quad \exp(\tilde{\mathbf{w}} + \tilde{\mathbf{h}}) = \exp(\mathbf{w}) \exp(\mathbf{h}),$$

where $\mathbf{w} \in \mathfrak{r}_1 + \mathfrak{r}_2$ with $\|\mathbf{w} - \tilde{\mathbf{w}}\| \leq 4\rho_2\|\tilde{\mathbf{w}}\|$, and $\mathbf{h} \in \mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$ with

$$\|\mathbf{h} - \tilde{\mathbf{h}}\| \leq 4\rho_1\|\tilde{\mathbf{h}}\|.$$

Proof. Let $\rho_0 = \max\{\rho_1, \rho_2\}$. By the Baker-Campbell-Hausdorff formula, we have

$$\exp(\mathbf{w})\exp(\mathbf{h}) = \exp(\mathbf{w} + \mathbf{h} + P(\mathbf{w}, \mathbf{h})),$$

where

$$P(\mathbf{w}, \mathbf{h}) = \frac{1}{2}[\mathbf{w}, \mathbf{h}] + \frac{1}{12}([\mathbf{w}, [\mathbf{w}, \mathbf{h}]] + [\mathbf{h}, [\mathbf{h}, \mathbf{w}]]) + \dots.$$

By Lemma 2.1 and the fact that for any $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{g}$, $\|[\mathbf{w}_1, \mathbf{w}_2]\| \leq \|\mathbf{w}_1\| \cdot \|\mathbf{w}_2\|$, we have that for any monomial $[\dots \diamond \dots]$ in $P(\mathbf{w}, \mathbf{h})$ with i Lie brackets, we have $[\dots \diamond \dots] \in \mathfrak{r}_1 + \mathfrak{r}_2$ or $[\dots \diamond \dots] \in \mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$, and

$$\|[\dots \diamond \dots]\| \leq \rho_0^{i-1}\|\mathbf{w}\| \cdot \|\mathbf{h}\|.$$

Therefore, we have that

$$P(\mathbf{w}, \mathbf{h}) = \bar{\mathbf{w}} + \bar{\mathbf{h}},$$

where $\bar{\mathbf{w}} \in \mathfrak{r}_1 + \mathfrak{r}_2$ with $\|\bar{\mathbf{w}}\| \leq 2\rho_2\|\mathbf{w}\|$, and $\bar{\mathbf{h}} \in \mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$ with $\|\bar{\mathbf{h}}\| \leq 2\rho_1\|\mathbf{h}\|$. This proves (2.5).

Conversely, for $\tilde{\mathbf{w}} \in B_{\mathfrak{r}_1+\mathfrak{r}_2}(\rho_1)$ and $\tilde{\mathbf{h}} \in B_{\mathbb{R}\mathfrak{a}_0+\mathfrak{h}}(\rho_2)$, let us write

$$\exp(\tilde{\mathbf{w}} + \tilde{\mathbf{h}}) = \exp(\mathbf{w})\exp(\mathbf{h}),$$

where $\mathbf{w} \in B_{\mathfrak{r}_1+\mathfrak{r}_2}(10^{-10})$ and $\mathbf{h} \in B_{\mathbb{R}\mathfrak{a}_0+\mathfrak{h}}(10^{-10})$. Then by (2.5), we have

$$\|\mathbf{w} - \tilde{\mathbf{w}}\| \leq 2\|\mathbf{h}\|\|\mathbf{w}\|, \text{ and } \|\mathbf{h} - \tilde{\mathbf{h}}\| \leq 2\|\mathbf{w}\|\|\mathbf{h}\|.$$

The above inequalities imply that $\|\mathbf{w}\| \leq 2\|\tilde{\mathbf{w}}\| \leq 2\rho_1$ and $\|\mathbf{h}\| \leq 2\|\tilde{\mathbf{h}}\| \leq 2\rho_2$. Therefore, we have

$$\|\mathbf{w} - \tilde{\mathbf{w}}\| \leq 4\rho_2\|\tilde{\mathbf{w}}\|, \text{ and } \|\mathbf{h} - \tilde{\mathbf{h}}\| \leq 4\rho_1\|\tilde{\mathbf{h}}\|.$$

This proves (2.6). $\square$

LEMMA 2.3. Let $0 < \rho < \rho_0 < 10^{-10}$ be two small constants. Let $\mathbf{w}, \mathbf{w}' \in B_{\mathfrak{r}_1+\mathfrak{r}_2}(\rho_0)$ be such that

$$\|[\mathbf{w}, \mathbf{w}']\| \leq \rho.$$

Then

$$\exp(\mathbf{w})\exp(\mathbf{w}') = \exp(\tilde{\mathbf{w}})\exp(\mathbf{h}),$$

where $\tilde{\mathbf{w}} \in \mathfrak{r}_1 + \mathfrak{r}_2$ with $\|\tilde{\mathbf{w}} - \mathbf{w} - \mathbf{w}'\| \leq 26\rho\rho_0$, and $\mathbf{h} \in B_{\mathbb{R}\mathfrak{a}_0+\mathfrak{h}}(4\rho)$.

Proof. By the Baker-Campbell-Hausdorff formula, we have

$$\exp(\mathbf{w})\exp(\mathbf{w}') = \exp(\mathbf{w} + \mathbf{w}' + P(\mathbf{w}, \mathbf{w}')),$$

where

$$P(\mathbf{w}, \mathbf{w}') = \frac{1}{2}[\mathbf{w}, \mathbf{w}'] + \frac{1}{12}([\mathbf{w}, [\mathbf{w}, \mathbf{w}']] + [\mathbf{w}', [\mathbf{w}', \mathbf{w}]]) + \dots.$$

By Lemma 2.1, we have the following observations:

For $i = 2j + 1$, every monomial $[\cdots \diamond \cdots]$ in $P(\mathbf{w}, \mathbf{w}')$ with i Lie brackets is in $\mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$. For $i = 2j$, every monomial $[\cdots \diamond \cdots]$ in $P(\mathbf{w}, \mathbf{w}')$ with i Lie brackets is in $\mathfrak{r}_1 + \mathfrak{r}_2$.

Moreover, by the condition $\|\mathbf{w}, \mathbf{w}'\| \leq \rho$ and the fact that for any $\mathbf{g}_1, \mathbf{g}_2 \in \mathfrak{g}$, $\|[\mathbf{g}_1, \mathbf{g}_2]\| \leq \|\mathbf{g}_1\| \cdot \|\mathbf{g}_2\|$, we have the following estimates:

For $i = 2j + 1$ and any monomial $[\cdots \diamond \cdots]$ with i Lie brackets,

$$\|[\cdots \diamond \cdots]\| \leq \rho \rho_0^{2j};$$

For $i = 2j$ and any $[\cdots \diamond \cdots]$ with i Lie brackets,

$$\|[\cdots \diamond \cdots]\| \leq \rho \rho_0^{2j-1}.$$

Thus, we have that $P(\mathbf{w}, \mathbf{w}') = \bar{\mathbf{w}} + \bar{\mathbf{h}}$, where $\bar{\mathbf{w}} \in \mathfrak{r}_1 + \mathfrak{r}_2$ with $\|\bar{\mathbf{w}}\| \leq 2\rho\rho_0$ and $\bar{\mathbf{h}} \in \mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$ with $\|\bar{\mathbf{h}}\| \leq 2\rho$. This implies that

$$\exp(\mathbf{w}) \exp(\mathbf{w}') = \exp(\hat{\mathbf{w}} + \bar{\mathbf{h}}),$$

where $\hat{\mathbf{w}} = \mathbf{w} + \mathbf{w}' + \bar{\mathbf{w}}$. In particular,

$$(2.7) \quad \|\hat{\mathbf{w}} - \mathbf{w} - \mathbf{w}'\| \leq 2\rho\rho_0.$$

It is easy to see that $\|\hat{\mathbf{w}}\| \leq 3\rho_0$. By Lemma 2.2, we have that

$$\exp(\hat{\mathbf{w}} + \bar{\mathbf{h}}) = \exp(\tilde{\mathbf{w}}) \exp(\mathbf{h}),$$

where $\tilde{\mathbf{w}} \in \mathfrak{r}_1 + \mathfrak{r}_2$ satisfies

$$(2.8) \quad \|\tilde{\mathbf{w}} - \hat{\mathbf{w}}\| \leq 8\rho\|\hat{\mathbf{w}}\| \leq 24\rho_0\rho,$$

and $\mathbf{h} \in \mathbb{R}\mathfrak{a}_0 + \mathfrak{h}$ with $\|\mathbf{h}\| \leq 4\rho$. Noting that (2.7) and (2.8) imply that

$$\|\tilde{\mathbf{w}} - \mathbf{w} - \mathbf{w}'\| \leq 26\rho_0\rho,$$

we conclude the lemma. $\square$

2.1. *Outline of the proof.* In this subsection we will give the outline of the proof.

The proof is inspired by Ratner's original proof of her measure rigidity theorem [Rat90a] and recent papers by Lindenstrauss, Mohammadi, and Wang [LM23] and [LMW23]. However, compared with previous works, in this paper we take a quite different approach.

In Section 6, we will start with $\mathcal{F}_0 = a(s)u([1])x_0$ for $s = t - \delta_1 t$ and study the dimension of the normalized Lebesgue measure $\mu_{\mathcal{F}_0}$ on $\mathcal{F}_0$ in directions transversal to the A_0H -orbit direction. We will show that $\mathcal{F}_0$ has certain dimension control in the transversal direction unless condition (2) in Theorem 1.3 holds (see Proposition 6.2). Here we are allowed to remove an exponentially small proportion from $\mathcal{F}_0$. At this step quantitative non-divergence results proved in Section 4 are needed. This is the starting point of our proof.

The argument in this part is similar to the corresponding parts in [LM23] and [LMW23].

Sections 7 to 11 are the crucial part. In these sections, we will construct a sequence $\{\mathcal{F}_i : i \in \mathbb{N}\}$ of U -orbits starting from $\mathcal{F}_0$, where $\mathcal{F}_{i+1} = a(s_i)\mathcal{F}_i$ for each $i \geq 0$. We will prove Proposition 9.1, which says that $\mu_{\mathcal{F}_{i+1}}$ has a better dimension control compared with $\mu_{\mathcal{F}_i}$ at a larger scale as a cost. To prove this, in Section 8 we introduce a Kekeya-type model to study the behavior of U -orbits in a neighborhood. The proof of Proposition 9.1 is divided into three parts: Proposition 9.9, Proposition 9.10 from Section 9, and Proposition 11.1 from Section 11. The key of the proofs is to calculate the weighted intersection number of the Kekeya-type model. The final outcome of the calculation is the following: Either $\mu_{\mathcal{F}_{i+1}}$ has a better dimension control, or the orbit is concentrated on a small neighborhood of a piece of an orbit of a subgroup isomorphic to $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ (see Lemma 11.8). The latter case can be ruled out by our Diophantine condition (see Proposition 6.3). At each step we are allowed to remove an exponentially small proportion. This part is novel compared with previous works.

The inductive construction given as above will conclude that for some $n > 0$, the dimension control of $\mu_{\mathcal{F}_n}$ along the directions transversal to the A_0H -orbit direction is almost optimal. Then we can apply Proposition 3.1 from Section 3 to conclude the effective equidistribution. Proposition 3.1 can be proved by following a Van der Corput type argument due to Venkatesh.

3. High transversal dimension to effective equidistribution

The following statement allows us to get effective equidistribution from high transversal dimension control. It can be proved by following Venkatesh's argument ([Ven10, §3] and [LM23, Proposition 4.2] for details).

PROPOSITION 3.1. *There exists $\theta > 0$ such that the following holds: For any $0 < \varepsilon < 10^{-1000}$, there exist $\eta > 0$ and $s_0 \gg 1$ depending on $\varepsilon > 0$ such that for any $s' \geq s_0$, with $e^{-\varepsilon^2 s'} \leq \beta \leq 10^{-10}$, any Borel probability measure ρ on $[\beta]^2$ with dimension larger than $2 - 2\theta$ at scale s' , that is, for any square $S \subset [\beta]^2$ of size $\beta e^{-s'}$, we have*

$$\rho(S) \leq (\beta e^{-s'})^{2-2\theta};$$

and for any function $f \in C_c^\infty(X)$ and any $x \in X_{\beta^{1/4}}$, we have

$$\left| \int_{[\beta]^2} \int_0^1 f(a(2s') \exp(v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1) u(\beta r) x) dr d\rho(w_1, v_1) - \int f d\mu_G \right| \ll e^{-\eta s'} \|f\|_S.$$

Proof. The statement can be proved by following the proof of [LM23, Proposition 4.2] step-by-step. $\square$

4. Quantitative non-divergence

In this section, we will prove a quantitative non-divergence result and a quantitative separation lemma, which will be used to prove [Proposition 6.2](#).

PROPOSITION 4.1. *There exists $\alpha > 0$ such that the following holds: For any $\beta, s > 0$, if $x \in X$ satisfies that $a_0(-s)x, a_0(s)x \notin X_{e^{-s/3}}$, then,*

$$\text{Leb}(\{r \in [1] : a(s)u(r)x \notin X_\beta\}) \ll \beta^\alpha.$$

Proof. Let us denote $x = g\Gamma$. By the Kleinbock–Margulis quantitative non-divergence theorem (cf. [\[Kle08, Theorem 2.2\]](#), [\[KM98, Theorem 5.2\]](#)), if the statement does not hold then there exists a nonzero $\mathbf{v} \in \bigwedge^j \mathbb{Z}^3$ (where $j = 1$ or 2) such that

$$\max_{r \in [1]} \|a(s)u(r)g\mathbf{v}\| \leq 1.$$

Case 1. $\mathbf{v} \in \mathbb{Z}^3$: Let us denote $\mathbf{v}' = a(s)g\mathbf{v} = (v'_1, v'_2, v'_3)$. Then

$$a(s)u(r)g\mathbf{v} = u(re^s)\mathbf{v}' = (v'_1, v'_2 + re^s v'_3, v'_3).$$

Then $\|a(s)u(r)g\mathbf{v}\| \leq 1$ implies that $|v'_1| \leq 1$, $|v'_2 + re^s v'_3| \leq 1$ for any $r \in [1]$. The latter easily implies that $|v'_2| \leq 1$ and $|v'_3| \leq e^{-s}$. This implies that $\|a_1(-s)\mathbf{v}'\| \leq e^{-s/3}$. Noting that $a_1(s) = a(s)a_0(s)$, we complete the proof for [Case 1](#).

Case 2. $\mathbf{v} \in \bigwedge^2 \mathbb{Z}^3$: Let us denote $\mathbf{v}' = a(s)g\mathbf{v} = (v'_1, v'_2, v'_3)$, where we use the coordinates with respect to the basis $\{\mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2\}$. Then

$$a(s)u(r)g\mathbf{v} = u(re^s)\mathbf{v}' = (v'_1, v'_2, v'_3 + re^s v'_2).$$

By repeating the same argument as in [Case 1](#), we have $|v'_1|, |v'_3| \leq 1$ and $|v'_2| \leq e^{-s}$, which implies that $\|b(-s)\mathbf{v}'\| \leq e^{-s/3}$. Noting that $b(s) = a(s)a_0(-s)$, we complete the proof for [Case 2](#). $\square$

Note that for any $\beta > 0$ small enough and any $x \in X_{\beta^{1/4}}$, $B^G(\beta)x$ embeds into X injectively.

We also need the following quantitative separation lemma:

LEMMA 4.2. *Suppose that for any $0 < \beta < 1$, any $\ell \geq 1$, and any $x \in X$ with $a(-\ell)x \in X_{\beta^{1/4}}$, the following holds:*

$$\exp(r_1 e^\ell \mathbf{u} + r_2 e^{\ell/2} \mathbf{v}_1 + r_3 e^{\ell/2} \mathbf{w}_1)x = \exp(\mathbf{v})x,$$

where $r_1, r_2, r_3 \in [\beta]$, $\|(r_1, r_2, r_3)\| \geq \beta/4$, and $\mathbf{v} \in B_{\mathfrak{g}}(\beta)$. Let us write

$$\mathbf{v} = u\mathbf{u} + u^*\mathbf{u}^* + a\mathbf{a} + a_0\mathbf{a}_0 + \sum_{j=1}^2 w_j \mathbf{w}_j + v_j \mathbf{v}_j.$$

If we have $|u^*| < \beta e^{-\ell}$, then $\|(v_2, w_2)\| \geq \beta e^{-\ell/2}$.

Proof. For a contradiction, let us assume that $|u^*| < \beta e^{-\ell}, |v_2, w_2| < \beta e^{-\ell/2}$. Let us denote $x = g\Gamma$. Then

$$\exp(r_1 e^\ell \mathbf{u} + r_2 e^{\ell/2} \mathbf{v}_1 + r_3 e^{\ell/2} \mathbf{w}_1)g = \exp(\mathbf{v})g\gamma,$$

for some $\gamma \in \Gamma$. For $\ell > 1$ large enough, we have $\gamma \neq e$. Then we have

$$\exp(-\mathbf{v})\exp(r_1 e^\ell \mathbf{u} + r_2 e^{\ell/2} \mathbf{v}_1 + r_3 e^{\ell/2} \mathbf{w}_1) = g\gamma g^{-1}.$$

Then

$$\begin{aligned} a(-\ell)g\gamma g^{-1}a(\ell) &= a(-\ell)\exp(-\mathbf{v})\exp(r_1 e^\ell \mathbf{u} + r_2 e^{\ell/2} \mathbf{w}_1)a(\ell) \\ &= \exp\left(-\text{Ad}(a(-\ell))\mathbf{v}\right)\exp(r_1 \mathbf{u} + r_2 \mathbf{v}_1 + r_3 \mathbf{w}_1). \end{aligned}$$

Since $|u^*| < \beta e^{-\ell}, |v_2, w_2| < \beta e^{-\ell/2}$, we have $\|\text{Ad}(a(-\ell))\mathbf{v}\| < \beta$. This implies that

$$a(-\ell)g\gamma g^{-1}a(\ell) \in B^G(\beta),$$

which means that $B^G(\beta)a(-\ell)x$ does not embed into X injectively. This contradicts the fact that $a(-\ell)x \in X_{\beta^{1/4}}$. This completes the proof. $\square$

5. List of constants

Here we list all constants that will be used in the arguments later.

- (1) $\delta_1 = \delta_2 = 10^{-5}$, $d_0 = 10^{-6}$, $C_1 = 10d_0^{-1} = 10^7$.
- (2) Let $\theta > 0$ be the constant from [Theorem 3.1](#). Without loss of generality, we can assume that $\theta < 10^{-10}$.
- (3) Let $\alpha > 0$ be the constant as in [Proposition 4.1](#), and let $\alpha_2 = \alpha/4$.
- (4) $\varepsilon_2 = 10^{-20}d_0\theta$, $N_1 = 10^5\varepsilon_2^{-1}$, $N_2 = 10^6$, $\varepsilon = e^{-10^5 N_1}$, $\beta = e^{-\varepsilon^3 t}$, and $\delta_3 = 10^{-5}\theta\varepsilon$.

6. Initial dimension control

6.1. *Initial construction of transversal sets.* We first introduce some notation.

For $\ell > 1$ and a subspace $V \subset \mathfrak{t}_1 + \mathfrak{t}_2$ generated by some vectors from the canonical basis of $\mathfrak{t}_1 + \mathfrak{t}_2$, namely, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2\}$, let us define

$$(6.1) \quad Q(V, \ell, \bar{\beta}) := \{\exp(\mathbf{w}) : \mathbf{w} \in B_{\mathfrak{t}_1 + \mathfrak{t}_2}(\bar{\beta}), \|p_V(\mathbf{w})\| \leq \bar{\beta}e^{-\ell}\},$$

where p_V denotes the projection to V with respect to the canonical basis.

Let us denote

$$(6.2) \quad \mathfrak{q}^{A_0 H}(\ell, \bar{\beta}) := \{\mathbf{h} \in B_{\mathbb{R}\mathfrak{a}_0 + \mathfrak{h}}(\bar{\beta}) : \|p_{\mathfrak{u}^*}(\mathbf{h})\| \leq \bar{\beta}e^{-\ell}\},$$

$$(6.3) \quad Q^{A_0 H}(\ell, \bar{\beta}) := \{h = \exp(\mathbf{h}) : \mathbf{h} \in \mathfrak{q}^{A_0 H}(\ell, \bar{\beta})\},$$

and

$$(6.4) \quad \mathfrak{r}^{A_0H}(\ell, \bar{\beta}) := \left\{ \mathbf{h} \in B_{\mathbb{R}\mathfrak{a}_0 + \mathfrak{h}}(\bar{\beta}) : \|p_{\mathbf{u}}(\mathbf{h})\| \leq \bar{\beta}e^{-\ell} \right\},$$

$$(6.5) \quad R^{A_0H}(\ell, \bar{\beta}) := \left\{ h = \exp(\mathbf{h}) : \mathbf{h} \in \mathfrak{r}^{A_0H}(\ell, \bar{\beta}) \right\}.$$

For $s_1, s_2 \geq 0$, $\bar{\beta} > 0$, let us denote

$$(6.6) \quad Q_{s_2}^{s_1}(\bar{\beta}) := \left\{ \exp(\mathbf{w}) : \mathbf{w} \in \mathfrak{r}_1 + \mathfrak{r}_2 : \|p_+(\mathbf{w})\| \leq \bar{\beta}e^{-s_1}, \|p_-(\mathbf{w})\| \leq \bar{\beta}e^{-s_2} \right\}$$

and $Q_s(\bar{\beta}) := Q_s^s(\bar{\beta})$.

Let $x_0 \in X$ be such that (2) in [Theorem 1.3](#) does not hold. Let us start with the normalized measure on $\mathcal{E} = a(s)u([1])x_0$ where $s = (1 - \delta_1)t > 0$. This is a U -orbit of length e^s . Let us define $\mathcal{F} \subset \mathcal{E}$ as follows:

Definition 6.1. The set $\mathcal{F} \subset \mathcal{E}$ is defined by removing from $\mathcal{E}$ points $y_0 = a(s)u(r)x_0 \in \mathcal{E}$ satisfying $a_0(\ell')a(-\ell)y_0 \notin X_{\beta^{1/4}}$ for some $0 \leq \ell \leq 2\delta_2 t$ and $-2\delta_1 t \leq \ell' \leq 2\delta_1 t$. Let $\mu_{\mathcal{F}}$ denote the normalized U -orbit measure on $\mathcal{F}$.

By [Proposition 4.1](#) and our assumption on x_{initial} , it is easy to show that the removed proportion is $O(s^2\beta^{\alpha_2})$. Therefore,

$$(6.7) \quad \mu_{\mathcal{E}} = \mu_{\mathcal{F}} + O(s^2\beta^{\alpha_2}),$$

where $\mu_{\mathcal{E}}$ denotes the normalized U -invariant measure on $\mathcal{E}$.

6.2. Initial dimension bound. We will prove the following proposition:

PROPOSITION 6.2. *For any $\ell \in [\delta_3 t, \delta_1 t]$ and any $x \in \mathcal{F}$, we have*

$$\mu_{\mathcal{F}} \left(Q_{\ell}(\beta) B^{A_0H}(\beta)x \right) \leq e^{-d_0\ell}.$$

Proof. Let $\eta_1 = 100d_0$. Let us divide $a(s)u([1])x_0$ into pieces of length $e^{\eta_1\ell}$.

It suffices to show that for each piece $\tilde{\mathcal{E}}$ from the above partition with $\tilde{\mathcal{F}} := \tilde{\mathcal{E}} \cap \mathcal{F} \neq \emptyset$ and any $x \in \tilde{\mathcal{F}}$, the following holds:

$$\left| \tilde{\mathcal{F}} \cap Q_{\ell}(\beta) B^{A_0H}(\beta)x \right| \leq e^{-d_0\ell} e^{\eta_1\ell}.$$

For a contradiction, let us assume that the statement does not hold for some $x \in \tilde{\mathcal{F}}$. Then we can find at least $e^{\eta_2\ell}$ (where $\eta_2 = \eta_1/4$) many $\bar{x} \in \tilde{\mathcal{F}} \cap Q_{\ell}(\beta) B^{A_0H}(\beta)x$, $r_{\bar{x}} \in [1]$ such that

$$u \left(r_{\bar{x}} e^{2d_0\ell} \right) \bar{x} = \exp(\mathbf{v}_{\bar{x}}) a_0^{\bar{x}} h_{\bar{x}} \bar{x},$$

where $a_0^{\bar{x}} \in B^{A_0}(\beta)$, $h_{\bar{x}} \in B^H(\beta)$, and $\mathbf{v}_{\bar{x}} \in B_{\mathfrak{r}_1 + \mathfrak{r}_2}(\beta e^{-\ell})$. Since $\bar{x}$ satisfies $\bar{x} \in X_{\beta^{1/4}}$ and $a(-2d_0\ell)\bar{x} \in X_{\beta^{1/4}}$, by [Lemma 4.2](#), if we write $h_{\bar{x}} = \exp(\mathbf{h}_{\bar{x}})$ and

$$\mathbf{h}_{\bar{x}} = a_0 \mathfrak{a} + a_1 \mathfrak{u} + a_2 \mathfrak{u}^*,$$

we will have

$$(6.8) \quad |a_0| \geq \beta e^{-2d_0\ell}.$$

Let us write $\bar{x} = u_{\bar{x}}x$, where $u_{\bar{x}} = u(re^{m_\ell})$ for some $r \in [1]$. Let us fix a representative $g \in G$ of x ; we then have

$$u(r_{\bar{x}}e^{d_0\ell})u_{\bar{x}}g = \exp(\mathbf{v}_{\bar{x}})a_0^{\bar{x}}h_{\bar{x}}u_{\bar{x}}g\gamma_{\bar{x}},$$

for some $\gamma_{\bar{x}} \in \Gamma$. The above equality is equivalent to

$$(6.9) \quad g\gamma_{\bar{x}}g^{-1} = (a_0^{\bar{x}}h_{\bar{x}}u_{\bar{x}})^{-1} \exp(-\mathbf{v}_{\bar{x}})u(r_{\bar{x}}e^{2d_0\ell})u_{\bar{x}}.$$

Without loss of generality, we can assume that for different $\bar{x}_1$ and $\bar{x}_2$ as above, $|u_{\bar{x}_1}^{-1}u_{\bar{x}_2}| \geq e^{5d_0\ell}$.

For each $\bar{x}$, we get $\gamma_{\bar{x}} \in \Gamma$. We claim that those $\gamma_{\bar{x}}$'s are different. In fact, if there exist $\bar{x}, \bar{x}'$ such that $\gamma_{\bar{x}} = \gamma_{\bar{x}'}$, then we have

$$\begin{aligned} & (a_0^{\bar{x}}h_{\bar{x}}u_{\bar{x}})^{-1} \exp(-\mathbf{v}_{\bar{x}})u(r_{\bar{x}}e^{2d_0\ell})u_{\bar{x}} \\ &= (a_0^{\bar{x}'}h_{\bar{x}'}u_{\bar{x}'})^{-1} \exp(-\mathbf{v}_{\bar{x}'})u(r_{\bar{x}'}e^{2d_0\ell})u_{\bar{x}'}. \end{aligned}$$

This implies that

$$\begin{aligned} & \exp(\mathbf{v}_{\bar{x}}') (a_0^{\bar{x}})^{-1} u_{\bar{x}}^{-1} h_{\bar{x}}^{-1} u_{\bar{x}} u (r_{\bar{x}}e^{2d_0\ell}) \\ &= \exp(\mathbf{v}_{\bar{x}'}') (a_0^{\bar{x}'})^{-1} u_{\bar{x}'}^{-1} h_{\bar{x}'}^{-1} u_{\bar{x}'} u (r_{\bar{x}'}e^{2d_0\ell}), \end{aligned}$$

where $\mathbf{v}_{\bar{x}}' = -\text{Ad}((a_0^{\bar{x}}h_{\bar{x}}u_{\bar{x}})^{-1})\mathbf{v}_{\bar{x}}$ and $\mathbf{v}_{\bar{x}'}'$ denotes the same expression with $\bar{x}$ replaced by $\bar{x}'$. By comparing the $\mathbf{r}_1 + \mathbf{r}_2$, A_0 and H components of both sides, we have $\exp(\mathbf{v}_{\bar{x}}') = \exp(\mathbf{v}_{\bar{x}'}')$, $a_0^{\bar{x}} = a_0^{\bar{x}'}$ and

$$u_{\bar{x}}^{-1}h_{\bar{x}}^{-1}u_{\bar{x}}u(r_{\bar{x}}e^{2d_0\ell}) = u_{\bar{x}'}^{-1}h_{\bar{x}'}^{-1}u_{\bar{x}'}u(r_{\bar{x}'}e^{2d_0\ell}).$$

This implies that

$$(6.10) \quad u(-L)h_{\bar{x}}u(L) = u(re^{2d_0\ell})h_{\bar{x}'},$$

where $u(L) = u_{\bar{x}}u_{\bar{x}'}^{-1}$ and $r = r_{\bar{x}} - r_{\bar{x}'}$. Note that $|r| \leq 1$ and $|L| \geq e^{10d_0\ell}$. The inequality $|r| \leq 1$ implies that the norm of the right-hand side of (6.10) does not exceed $e^{2d_0\ell}$. On the other hand, the left-hand side is equal to

$$\exp(\text{Ad}(u(-L))\mathbf{h}_{\bar{x}}) = \exp(\text{Ad}(u(-L))(a_0\mathbf{a} + a_1\mathbf{u} + a_2\mathbf{u}^*))$$

whose $\mathbf{u}$ coordinate is $a_1 + a_0L + a_2L^2$. Since $|a_2| \geq \beta e^{-2d_0\ell}$ (by (6.8)) and $|L| \geq e^{5d_0\ell}$, we have that

$$|a_1 + a_0L + a_2L^2| \geq \beta e^{8d_0\ell}.$$

Therefore, we have that the equality (6.10) is impossible to hold and conclude the claim.

Let us consider the adjoint action of G on $\bigoplus_{j=1}^8 \wedge^j \mathfrak{g}$ and denote

$$v_H = \mathbf{a} \wedge \mathbf{u} \wedge \mathbf{u}^*.$$

Then the stabilizer of v_H is as follows:

$$\text{Stab}(v_H) = A_0 H.$$

Then (6.9) implies that

$$\gamma_{\bar{x}} g^{-1} v_H = \exp(\text{Ad}(g^{-1}) \mathbf{v}'_{\bar{x}}) g^{-1} v_H.$$

The norm of $g^{-1} v_H$ is $\leq \|g^{-1}\|^{300}$. Let us estimate the norm of $\mathbf{v}'_{\bar{x}}$. Note that $\|u_{\bar{x}}\| \leq e^{\eta_1 \ell}$, we have that the norm of $(a_0^{\bar{x}} h_{\bar{x}} u_{\bar{x}})^{-1}$ is bounded by $2e^{\eta_1 \ell}$. Therefore, $\|\mathbf{v}'_{\bar{x}}\| \leq e^{-\ell + \eta_1 \ell}$. Let us denote $v_0 = g^{-1} v_H / \|g^{-1} v_H\|$; then we have

$$\|\gamma_{\bar{x}} v_0 - v_0\| \leq \|\mathbf{v}'_{\bar{x}}\| \leq e^{-(1-\eta_1)\ell}.$$

Considering the group generated by $\gamma_{\bar{x}}$'s, we have the following two cases:

- (1) $\langle \gamma_{\bar{x}} \rangle$ is abelian.
- (2) $\langle \gamma_{\bar{x}} \rangle$ is not abelian.

Case 1. The group $\langle \gamma_{\bar{x}} \rangle$ is abelian. We claim that there exists $\gamma_{\bar{x}}$ whose unipotent part is not trivial. In fact, if every $\gamma_{\bar{x}}$ is diagonalizable, then they belong to a maximal torus in G . Note that $\|\gamma_{\bar{x}}\| \leq e^{\eta_1 \ell}$; we have that there are at most ℓ^{100} different $\gamma_{\bar{x}}$'s that contradicts the fact that there are $e^{\eta_2 \ell}$ different $\gamma_{\bar{x}}$'s. Note that $\gamma_{\bar{x}} \in \text{SL}(3, \mathbb{Z})$; if $\gamma_{\bar{x}}$ has a nontrivial unipotent part, then it must be unipotent. By repeating the same argument as in [EMV09], we can find $g' \in G$ satisfying the following: $\gamma_{\bar{x}} g'^{-1} v_H = g'^{-1} v_H$ and

$$\|g' - g\| \leq \|\gamma_{\bar{x}} v_0 - v_0\| \|\gamma_{\bar{x}}\|^{10} \leq e^{-(1-\eta_1)\ell} e^{10\eta_1 \ell} \leq e^{-\ell/2}.$$

Then $g' \gamma_{\bar{x}} g'^{-1} \in A_1 H$. Since $\gamma_{\bar{x}}$ is unipotent, we have $g' \gamma_{\bar{x}} g'^{-1} \in H$. We claim that the lattice $g' \mathbb{Z}^3$ contains a nonzero vector $\mathbf{p}$ satisfying $\|\mathbf{p}\| \leq e^{2\eta_1 \ell}$ and $\mathbf{p} \in \mathbb{R} \mathbf{e}_2 + \mathbb{R} \mathbf{e}_3$. In fact, we can find a basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ of $g' \mathbb{Z}^3$ such that $\|\mathbf{p}_j\| \leq \|g'\| \leq 2\|g\| \leq 2\beta^{-1}$ for $j = 1, 2, 3$. Since $g' \gamma_{\bar{x}} g'^{-1}$ is a unipotent element in H , its fixing vectors

$$V(g' \gamma_{\bar{x}} g'^{-1}) := \{\mathbf{p} \in \mathbb{R}^3 : g' \gamma_{\bar{x}} g'^{-1} \mathbf{p} = \mathbf{p}\}$$

have dimension 2. Therefore, $g' \gamma_{\bar{x}} g'^{-1} \mathbf{p}_j \neq \mathbf{p}_j$ for some $j \in \{1, 2, 3\}$. On the other hand, we have $g' \gamma_{\bar{x}} g'^{-1} \mathbf{p}_j \in g' \mathbb{Z}^3$. Thus,

$$g' \gamma_{\bar{x}} g'^{-1} \mathbf{p}_j - \mathbf{p}_j \in g' \mathbb{Z}^3.$$

Moreover, since $g' \gamma_{\bar{x}} g'^{-1} \in H$, it is easy to see that $\mathbf{p} := g' \gamma_{\bar{x}} g'^{-1} \mathbf{p}_j - \mathbf{p}_j \in \mathbb{R} \mathbf{e}_2 + \mathbb{R} \mathbf{e}_3$. Its norm is bounded by

$$\begin{aligned} \|\mathbf{p}\| &\leq \|\mathbf{p}_j\| + \|g' \gamma_{\bar{x}} g'^{-1} \mathbf{p}_j - \mathbf{p}_j\| \\ &\leq \|\mathbf{p}_j\| + \|g'\| \|\gamma_{\bar{x}}\| \|g'^{-1} \mathbf{p}_j\| \leq 2\beta^{-1} + e^{\eta_1 \ell} \beta^{-10} \leq e^{2\eta_1 \ell}. \end{aligned}$$

Then for any $\eta_3 > 0$,

$$\|a_0(\eta_3 \ell) \mathbf{p}\| = e^{-\eta_3 \ell / 6} \|\mathbf{p}\| \leq e^{-(\eta_3 / 6 - 2\eta_1) \ell}.$$

Let us choose $\eta_3 = 24\eta_1$. Then it is easy to see that

$$\|a_0(\eta_3\ell)g' - a_0(\eta_3\ell)g\| \leq 1,$$

and $a_0(\eta_3\ell)g'\Gamma \notin X_{e^{-2\eta_1\ell}}$. This implies that $a_0(\eta_3\ell)g\Gamma \notin X_{\beta^{1/4}}$, which contradicts the fact that $g\Gamma = x \in \mathcal{F}$.

Case 2. The group $\langle \gamma_{\bar{x}} \rangle$ is not abelian. Let us take $\gamma = \gamma_{\bar{x}}$ and $\gamma' = \gamma_{\bar{x}'}$ not commuting. Using the same argument as in [Case 1](#) (again using the argument in [\[EMV09\]](#)), we can find $g' \in G$ satisfying the following: $\gamma g'^{-1}v_H = g'^{-1}v_H$, $\gamma' g'^{-1}v_H = g'^{-1}v_H$ and

$$\|g' - g\| \leq e^{-\ell/2}.$$

This implies that $g'\gamma g'^{-1}, g'\gamma' g'^{-1} \in A_1H$. Then their commutator

$$g'\gamma\gamma'\gamma^{-1}\gamma'^{-1}g'^{-1} \in H.$$

Then we can use the same argument as in [Case 1](#) to deduce that $a_0(\eta_3\ell)g\Gamma \notin X_{\beta^{1/4}}$ leading to the same contradiction.

This completes the proof of [Proposition 6.2](#). $\square$

The following statement tells that the orbit can not be concentrated on a small neighborhood of a piece of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ orbit:

PROPOSITION 6.3. *For any $x \in \mathcal{F}$ and any $\ell \in [2\delta_3 t, \delta_1 t]$ with $j = 1, 2$, the following holds:*

$$\mu_{\mathcal{F}}(Q(\mathbf{r}_j, \ell, \beta)B^{A_0H}(\beta)x) \leq e^{-d_0\ell/2}.$$

Proof. Let us prove the statement for $j = 1$. The proof for $j = 2$ is similar. For a contradiction, let us assume that there exists $x \in \mathcal{F}$ such that

$$\mu_{\mathcal{F}}(Q(\mathbf{r}_1, \ell, \beta)B^{A_0H}(\beta)x) > e^{-d_0\ell/2}.$$

Let us denote $\tilde{x} = a_0(\ell)x$ and $\hat{\mu} = a_0(\ell)_*\mu_{\mathcal{F}}$. Noting that

$$a_0(\ell)Q(\mathbf{r}_1, \ell, \beta)B^{A_0H}(\beta)x = Q_{\ell/2}(\beta)B^{A_0H}(\beta)\tilde{x},$$

we have that

$$\hat{\mu}(Q_{\tilde{\ell}}(\beta)B^{A_0H}(\beta)\tilde{x}) > e^{-d_0\tilde{\ell}},$$

where $\tilde{\ell} = \ell/2 \in [\delta_3 t, \delta_1 t]$. By repeating the argument in the proof of [Proposition 6.2](#), we can show that $a_0(\eta_3\tilde{\ell})\tilde{x} = a_0(\ell + \eta_3\tilde{\ell})x \notin X_{\beta^{1/4}}$, which contradicts the fact that $x \in \mathcal{F}$. This completes the proof. $\square$

7. Basic reduction

In this section, we introduce some notation and make a basic reduction of the original problem.

Let us first introduce some definitions:

Definition 7.1. Given a probability measure μ on X , we say that μ is (d, ℓ) -good if, for any $x_0 \in X$, the following holds:

$$\mu\left(Q_\ell(\beta)B^{A_0H}(\beta)x_0\right) \leq e^{-d\ell}.$$

We say that μ is $(d, [\ell_1, \ell_2])$ -good if it is (d, ℓ) -good for any $\ell \in [\ell_1, \ell_2]$.

According to this definition, [Proposition 6.2](#) says that $\mu_{\mathcal{F}}$ is $(d_0, [\delta_3 t, \delta_1 t])$ -good.

Starting from $s_0 = \delta_1 t/2$ and d_0 , let us define $\{(d_i, s_i) : i \in \mathbb{N}\}$ by $s_{i+1} = s_i/2$ and $d_{i+1} = d_i + \varepsilon_2$.

Starting from $\mathcal{F}_0 = \mathcal{F}$, we will construct a sequence $\{\mathcal{F}_i : i \in \mathbb{N}\}$ such that every $\mathcal{F}_{i+1}$ is constructed by removing a small proportion from $a(s_i)\mathcal{F}_i$.

In the following several sections, we will prove that $\mu_{\mathcal{F}_{i+1}}$ has better dimension control than $\mu_{\mathcal{F}_i}$. Similarly to [Definition 6.1](#), by removing an exponentially small proportion, we can assume that for any $x \in \text{supp}\mu_{\mathcal{F}_i}$, any $0 \leq \ell \leq 2\delta_2 t$ and $-2\delta_1 t \leq \ell' \leq 2\delta_1 t$, the following holds: $a_0(\ell')a(-\ell)x \in X_{\beta^{1/4}}$. By repeating the proof of [Propositions 6.2](#) and [6.3](#), we have that [Propositions 6.2](#) and [6.3](#) hold for $\mu_{\mathcal{F}_i}$.

For notational simplicity, for $s' > 0, \ell \geq 0$, let us denote

$$\Pi_{s', \ell} := a(2s')B^{A_0A}(\beta)B^{U^*}(\beta)\exp\left(B_{\mathbf{r}^-}(\beta e^{-\ell})\right)\exp\left(B_{\mathbf{r}^+}(\beta e^{-\ell})\right)B^U(\beta)a(-2s')$$

and

$$(7.1) \quad \Pi_{s'} := \Pi_{s', 0}.$$

For $x \in X$, let us denote

$$\Pi(x) := \Pi_{s'}x.$$

Given that $\mu_{\mathcal{F}_i}$ is (d_i, s_i) -good, our aim is to prove that $\mu = a(s_i)_*\mu_{\mathcal{F}_i}$ is $(d_i + \varepsilon_2, s_{i+1})$ -good after removing an exponentially small proportion.

We have the following reduction of the original task:

PROPOSITION 7.2. *For $x \in X$, let us call a neighborhood of the form $\Pi(x) = \Pi_{s'}x$ a playground. A playground $\Pi(x)$ is called a full playground if*

$$\mu(\Pi(x)) \geq \beta^{15}.$$

For any full playground $\Pi(x)$, let us denote

$$\mu_{\Pi(x)} := \mu(\Pi(x))^{-1}\mu|_{\Pi(x)}.$$

If, for any full playground $\Pi(x)$, there exists $\mathcal{R}_x \subset \Pi(x)$ such that

$$\mu_{\Pi(x)}(\mathcal{R}_x) \leq \beta^{20},$$

and $\mu_{\Pi(x)}$ is $(d_i + 2\varepsilon_2, s_{i+1})$ -good after removing $\mathcal{R}_x$, then there exists $\mathcal{R}_\diamond \subset X$ such that

$$\mu(\mathcal{R}_\diamond) \leq \beta^2,$$

and μ is $(d_i + \varepsilon_2, s_{i+1})$ -good after removing $\mathcal{R}_\diamond$.

Proof. Note that there exists a finite subset $\mathfrak{Q} \subset X$ with $|\mathfrak{Q}| \leq \beta^{-9}$ and

$$X = \bigcup_{x \in \mathfrak{Q}} \Pi(x).$$

Let $\mathfrak{Q}_1 \subset \mathfrak{Q}$ denote the set of $x \in \mathfrak{Q}$ such that $\Pi(x)$ is a full playground, and $\mathfrak{Q}_2 = \mathfrak{Q} \setminus \mathfrak{Q}_1$. Then

$$\mu \left(\bigcup_{x \in \mathfrak{Q}_2} \Pi(x) \right) \leq \beta^{-9} \times \beta^{15} \leq \beta^6.$$

By the given condition, for every $x \in \mathfrak{Q}_1$, we get $\mathcal{R}_x \subset \Pi(x)$ with

$$\mu_{\Pi(x)}(\mathcal{R}_x) \leq \beta^{20},$$

and $\mu_{\Pi(x)}$ is $(d_i + 2\varepsilon_2, s_{i+1})$ -good after removing $\mathcal{R}_x$. Then

$$\mu \left(\bigcup_{x \in \mathfrak{Q}_1} \mathcal{R}_x \right) \leq \beta^{-9} \times \beta^{20} = \beta^{11}.$$

Let

$$\mathcal{R}_\diamond = \bigcup_{x \in \mathfrak{Q}_1} \mathcal{R}_x \cup \bigcup_{x \in \mathfrak{Q}_2} \Pi(x).$$

Then $\mu(\mathcal{R}_\diamond) \leq \beta^6 + \beta^{11} \leq \beta^2$.

Let μ' denote the measure constructed by removing $\mathcal{R}_\diamond$ from μ . Let us prove that μ' is $(d_i + \varepsilon_2, s_{i+1})$ -good. In fact, for any $x' \in X$, let us denote

$$Q(x') = Q_{s_{i+1}}(\beta) B^{A_0 H}(\beta) x'.$$

Then we have

$$\mu'(Q(x')) \leq \sum_{x \in \mathfrak{Q}} \mu'_{\Pi(x)}(Q(x')),$$

where $\mu'_{\Pi(x)}$ denotes the measure constructed by removing $\mathcal{R}_\diamond$ from $\mu_{\Pi(x)}$. Note that for any $x \in \mathfrak{Q}$, $\mu'_{\Pi(x)}$ is $(d_i + 2\varepsilon_2, s_{i+1})$ -good; we have

$$\mu'_{\Pi(x)}(Q(x')) \leq e^{-(d_i + 2\varepsilon_2)s_{i+1}}.$$

Thus,

$$\mu'(Q(x')) \leq \beta^{-9} \times e^{-(d_i + 2\varepsilon_2)s_{i+1}} \leq e^{-(d_i + \varepsilon_2)s_{i+1}}.$$

This completes the proof. $\square$

Let us denote $\varepsilon_1 = 2\varepsilon_2$. By [Proposition 7.2](#), it suffices to prove that for any full playground $\Pi(x)$, the measure $\mu_{\Pi(x)}$ is $(d_i + \varepsilon_1, s_{i+1})$ -good after removing a β^{20} -small proportion. Therefore, in the following several sections, we will focus on a fixed full playground $\Pi(x)$ and study the measure $\tilde{\mu} = \mu_{\Pi(x)}$ on $\Pi(x)$.

8. Construction of a Kekeya-type model

From now on we will fix a full playground $\Pi(x)$ and denote $\tilde{\mu} = \mu_{\Pi(x)}$.

In this section we will introduce a Kekeya-type model, which enables us to study the dimension of the orbit measure.

Let us introduce some notation.

Let us denote

$$(8.1) \quad \Omega_{s'} := Q_{3s'}^{s'}(2\beta)Q^{A_0H}(2s', 2\beta),$$

and

$$\Theta_{s'} := B^{A_0A}(\beta)B^{U^*}(\beta e^{-2s'}) \exp(B_{\mathbf{r}^-}(\beta e^{-s'})) \exp(B_{\mathbf{r}^+}(\beta e^{-s'})) B^U(\beta).$$

For $s' > 0$, let us denote

$$(8.2) \quad \Sigma_{s'} := B^{A_0A}(\beta)B^{U^*}(\beta e^{-2s'})B_{\mathbf{r}^-}(\beta e^{-s'})B_{\mathbf{r}^+}(\beta e^{s'})B^U(\beta).$$

For $x \in X$, let us denote

$$\Sigma(x) := \Sigma_{s'}x,$$

$$\Theta(x) := \Theta_{s'}x,$$

and

$$\Omega(x) := \Omega_{s'}x.$$

We will need the following lemmata:

LEMMA 8.1. *Let us denote*

$$\hat{\Theta}_{s'} := Q_{s'}(1.1\beta)Q^{A_0H}(2s', 1.1\beta).$$

Then

$$\Theta_{s'} \subset \hat{\Theta}_{s'}.$$

Proof. Let us take

$$g = a_0au^*(r^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+)u(r) \in \Theta_{s'}.$$

By [Lemma 2.3](#),

$$\exp(\mathbf{w}^-) \exp(\mathbf{w}^+) = \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}),$$

where $\bar{\mathbf{w}} \in B_{\mathbf{r}_1 + \mathbf{r}_2}(1.01\beta e^{-s'})$ and $\bar{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathfrak{h}}(0.01\beta e^{-2s'})$. Therefore, we have

$$g = a_0au^*(r^*) \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}})u(r).$$

By rearranging the equation above, we have that $g \in \hat{\Theta}_{s'}$. $\square$

LEMMA 8.2. *Let us denote*

$$\bar{\Theta}_{s'} := Q_{s'}(2.21\beta)Q^{A_0H}(2s', 2.21\beta).$$

Then for any $\varepsilon_i \in \{\pm 1\}$ with $i = 1, 2$,

$$\Theta_{s'}^{\varepsilon_1} \Theta_{s'}^{\varepsilon_2} \subset \bar{\Theta}_{s'}.$$

Proof. We will only prove

$$\Theta_{s'}^{-1} \Theta_{s'} \subset \bar{\Theta}_{s'}.$$

Other cases can be proved similarly.

By Lemma 8.1, it suffices to show that

$$\hat{\Theta}_{s'}^{-1} \hat{\Theta}_{s'} \subset \bar{\Theta}_{s'},$$

where $\hat{\Theta}_{s'}$ is defined as in Lemma 8.1.

Let us take $g = \exp(\mathbf{w}) \exp(\mathbf{h}) \in \hat{\Theta}_{s'}$ and $g' = \exp(\mathbf{w}') \exp(\mathbf{h}') \in \hat{\Theta}_{s'}$. Then

$$g^{-1}g' = \exp(-\mathbf{h}) \exp(-\mathbf{w}) \exp(\mathbf{w}') \exp(\mathbf{h}').$$

By Lemma 2.3, we have

$$\exp(-\mathbf{w}) \exp(\mathbf{w}') = \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}),$$

where $\bar{\mathbf{w}} \in B_{\mathfrak{r}_1 + \mathfrak{r}_2}(2.201\beta e^{-s'})$, and $\bar{\mathbf{h}} \in B_{\mathbb{R}\mathfrak{a}_0 + \mathfrak{h}}(0.001\beta e^{-2s'})$. Thus, we have

$$g^{-1}g' = \exp(-\mathbf{h}) \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}) \exp(\mathbf{h}').$$

By rearranging the equation above, we conclude that $g^{-1}g' \in \bar{\Theta}_{s'}$. This completes the proof. $\square$

Using Lemma 8.2, it is easy to prove the following:

COROLLARY 8.3. *Let us denote*

$$\tilde{\Theta}_{s'} := Q_{s'}(4.5\beta)Q^{A_0H}(2s', 4.5\beta).$$

Then for any $\varepsilon_i \in \{\pm 1\}$ with $i = 1, 2, 3, 4$,

$$\Theta_{s'}^{\varepsilon_1} \Theta_{s'}^{\varepsilon_2} \Theta_{s'}^{\varepsilon_3} \Theta_{s'}^{\varepsilon_4} \subset \tilde{\Theta}_{s'}.$$

Proof. We will only prove $\Theta_{s'} \Theta_{s'} \Theta_{s'} \Theta_{s'} \subset \tilde{\Theta}_{s'}$. Other cases can be proved similarly. In fact, by Lemma 8.2, we have

$$\Theta_{s'} \Theta_{s'} \subset \bar{\Theta}_{s'},$$

where $\bar{\Theta}_{s'}$ is defined as in Lemma 8.2. By repeating the proof of Lemma 8.2, we get

$$\bar{\Theta}_{s'} \bar{\Theta}_{s'} \subset \tilde{\Theta}_{s'}.$$

Combining the two inclusions above, we conclude that

$$\Theta_{s'} \Theta_{s'} \Theta_{s'} \Theta_{s'} \subset \tilde{\Theta}_{s'}.$$

$\square$

We will need the following lemma concerning different coordinate systems in G :

LEMMA 8.4. *For $\mathbf{w}^+ \in \mathbf{r}^+$ and $\mathbf{w}^- \in \mathbf{r}^-$ with $\|\mathbf{w}^+\|, \|\mathbf{w}^-\| \leq 10\beta e^{-\ell}$, we have that*

$$\exp(\mathbf{w}^+ + \mathbf{w}^-) = a_0 a u^*(r^*) \exp(\bar{\mathbf{w}}^-) \exp(\bar{\mathbf{w}}^+) u(r),$$

where $\bar{\mathbf{w}}^+ \in \mathbf{r}^+$ with $\|\bar{\mathbf{w}}^+ - \mathbf{w}^+\| \leq 0.01\beta e^{-3\ell}$, $\bar{\mathbf{w}}^- \in \mathbf{r}^-$ with $\|\bar{\mathbf{w}}^- - \mathbf{w}^-\| \leq 0.01\beta e^{-3\ell}$, $a_0 a \in B^{A_0 A}(0.01\beta e^{-2\ell})$, and $r^*, r \in [0.01\beta e^{-2\ell}]$.

Similarly, we have

$$\exp(\mathbf{w}^+ + \mathbf{w}^-) = a_0 a u(r) \exp(\bar{\mathbf{w}}^+) \exp(\bar{\mathbf{w}}^-) u^*(r^*),$$

where $\bar{\mathbf{w}}^+ \in \mathbf{r}^+$ with $\|\bar{\mathbf{w}}^+ - \mathbf{w}^+\| \leq 0.01\beta e^{-3\ell}$, $\bar{\mathbf{w}}^- \in \mathbf{r}^-$ with $\|\bar{\mathbf{w}}^- - \mathbf{w}^-\| \leq 0.01\beta e^{-3\ell}$, $a_0 a \in B^{A_0 A}(0.01\beta e^{-2\ell})$, and $r^*, r \in [0.01\beta e^{-2\ell}]$.

Proof. We only prove the first statement. The second statement can be proved using the same argument.

Let us write

$$\exp(\mathbf{w}^+ + \mathbf{w}^-) = a_0 a u^*(r^*) \exp(\bar{\mathbf{w}}^-) \exp(\bar{\mathbf{w}}^+) u(r),$$

where $a_0 a \in B^{A_0 A}(10\beta e^{-\ell})$, $r^*, r \in [10\beta e^{-\ell}]$, $\bar{\mathbf{w}}^- \in B_{\mathbf{r}^-}(10\beta e^{-\ell})$, and $\bar{\mathbf{w}}^+ \in B_{\mathbf{r}^+}(10\beta e^{-\ell})$. Noting that

$$\|[\bar{\mathbf{w}}^-, \bar{\mathbf{w}}^+]\| \leq \|\bar{\mathbf{w}}^-\| \cdot \|\bar{\mathbf{w}}^+\| \leq 0.001\beta e^{-2\ell},$$

by Lemma 2.3, we have

$$\exp(\bar{\mathbf{w}}^-) \exp(\bar{\mathbf{w}}^+) = \exp(\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+) \exp(\mathbf{h}),$$

where $\tilde{\mathbf{w}}^- \in \mathbf{r}^-$ with $\|\tilde{\mathbf{w}}^- - \bar{\mathbf{w}}^-\| \leq 0.002\beta e^{-3\ell}$, $\tilde{\mathbf{w}}^+ \in \mathbf{r}^+$ with $\|\tilde{\mathbf{w}}^+ - \bar{\mathbf{w}}^+\| \leq 0.002\beta e^{-3\ell}$, and $\mathbf{h} \in B_{\mathbb{R}a_0 + \mathfrak{h}}(0.002\beta e^{-2\ell})$. Thus, we get

$$\begin{aligned} \exp(\mathbf{w}^+ + \mathbf{w}^-) &= a_0 a u^*(r^*) \exp(\bar{\mathbf{w}}^-) \exp(\bar{\mathbf{w}}^+) u(r) \\ &= a_0 a u^*(r^*) \exp(\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+) \exp(\mathbf{h}) u(r) \\ &= \exp(\text{Ad}_h(\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+)) a_0 a u^* \exp(\mathbf{h}) u(r), \end{aligned}$$

where $h = a_0 a u^*(r^*)$. Then we have

$$\mathbf{w}^+ + \mathbf{w}^- = \text{Ad}_h(\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+) \quad \text{and} \quad a_0 a u^*(r^*) \exp(\mathbf{h}) u(r) = \text{id}.$$

Note that $\|\mathbf{h}\| \leq 0.003\beta e^{-2\ell}$; we get $a_0 a \in B^{A_0 A}(0.01\beta e^{-2\ell})$, $r^*, r \in [0.01\beta e^{-2\ell}]$.

Note also that

$$\|\text{Ad}_h(\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+) - (\tilde{\mathbf{w}}^- + \tilde{\mathbf{w}}^+)\| \leq 0.001\beta e^{-3\ell}.$$

Combining the above estimates, we get

$$\|\mathbf{w}^+ - \bar{\mathbf{w}}^+\|, \|\mathbf{w}^- - \bar{\mathbf{w}}^-\| \leq 0.01\beta e^{-3\ell}.$$

This completes the proof. $\square$

For $r \in [\beta e^{2s'}]$, let us denote

$$(8.3) \quad \Sigma^r = \Sigma(u(r)x) \subset \Pi(x).$$

Σ^r can be regarded as a cross-section of $\Pi(x)$ along the U -orbits in the following sense:

LEMMA 8.5. *For $y \in \Pi(x)$ and $r \in [\beta e^{2s'}]$, let*

$$J_r(y) := \left\{ r' \in [2\beta e^{2s'}] : u(r')y \in \Sigma^r \right\}.$$

Then there exist intervals $J_1 \subset J_r(y) \subset J_2$ with size $0.5\beta \leq |J_1| \leq |J_2| \leq 4\beta$.

We will postpone the proof until after [Definition 8.7](#).

For $\bar{x} \in \Sigma^r \subset \Pi(x)$, let us denote

$$\Pi_{s',\ell}(\bar{x}) = \Pi_{s',\ell}\bar{x},$$

and

$$\Sigma_{s',\ell}(\bar{x}) = \Pi_{s',\ell}(\bar{x}) \cap \Sigma^r.$$

Then $\Sigma_{s',\ell}(\bar{x})$ can also be regarded as a cross-section of $\Pi_{s',\ell}(\bar{x})$ along the U -orbits.

We first prove the following lemma:

LEMMA 8.6. *For any $\delta_3 t \leq \ell \leq \delta_1 t$ and any $\bar{x} \in \Pi(x)$, we have*

$$\tilde{\mu}(\Sigma_{s',\ell}(\bar{x})) \leq e^{-2s'} e^{-d_0 \ell}.$$

Proof. Note that $\Sigma_{s',\ell}(\bar{x})$ is a cross-section of $\Pi_{s',\ell}(\bar{x})$ along the U -orbits, and $\tilde{\mu}$ is Lebesgue along the U -orbits. We have

$$\tilde{\mu}(\Sigma_{s',\ell}(\bar{x})) \ll e^{-2s'} \tilde{\mu}(\Pi_{s',\ell}(\bar{x})).$$

Note that by the definition of $\tilde{\mu}$, we have

$$\tilde{\mu}(\Pi_{s',\ell}(\bar{x})) \leq \mu_{\mathcal{F}_i}(a(-2s')\Pi_{s',\ell}(\bar{x})),$$

and

$$a(-2s')\Pi_{s',\ell}(\bar{x}) = B^{A_0 A}(\beta) B^{U^*}(\beta) \exp(B_{\mathbf{r}^-}(\beta e^{-\ell})) \exp(B_{\mathbf{r}^+}(\beta e^{-\ell})) B^U(\beta) x'$$

where $x' = a(-2s')\bar{x}$. By [Lemma 8.4](#), we have that

$$B^{A_0 A}(\beta) B^{U^*}(\beta) \exp(B_{\mathbf{r}^-}(\beta e^{-\ell})) \exp(B_{\mathbf{r}^+}(\beta e^{-\ell})) B^U(\beta) \subset Q_\ell(2\beta) B^{A_0 H}(2\beta).$$

By applying the proof of [Proposition 6.2](#) to $\mathcal{F}_i$ we have

$$\mu_{\mathcal{F}_i}(Q_\ell(2\beta) B^{A_0 H}(2\beta) x') \leq e^{-d_0 \ell},$$

which concludes the proof. $\square$

We then introduce the Kakeya-type model to analyze the behavior of U -orbits.

Definition 8.7. Let s' be a positive number, and let $\tilde{\mu}$ be a probability measure on a playground $\Pi(x) \subset X$ defined by taking the normalized measure on a subset consisting of U -orbits of length $\beta e^{2s'}$. Let us fix a sequence of cross-sections

$$\left\{ \Sigma^r = \Sigma_{s'} u(r)x : r \in [\beta e^{2s'}] \cap (0.1\beta\mathbb{Z}) \right\}.$$

We first define the projection map

$$\mathbb{P} : \Pi(x) \rightarrow \mathbb{R}^3$$

as follows: For $x' \in \Pi(x)$, we can write

$$x' = a_{0,x'} a_{x'} u^*(r_{x'}^*) \exp(\mathbf{w}_{x'}^-) \exp(\mathbf{w}_{x'}^+) u(r_{x'}) x,$$

where $a_{0,x'} \in B^{A_0}(\beta)$, $a_{x'} \in B^A(\beta)$, $\mathbf{w}_{x'}^- = v_{2,x'} \mathbf{v}_2 + w_{2,x'} \mathbf{w}_2 \in \mathbf{r}^-$ and $\mathbf{w}_{x'}^+ = v_{1,x'} \mathbf{v}_1 + w_{1,x'} \mathbf{w}_1 \in \mathbf{r}^+$, $r_{x'}^* \in [\beta e^{-2s'}]$, and $r_{x'} \in [\beta e^{2s'}]$. Then

$$\mathbb{P}(x') := (r_{x'}, v_{1,x'}, w_{1,x'}).$$

Then

$$\mathbb{P}(\Pi(x)) = [\beta e^{2s'}] \times [\beta e^{s'}]^2 \subset \mathbb{R}^3.$$

It is easy to see that for $r \in [\beta e^{2s'}]$,

$$\mathbb{P}(\Sigma^r) = (r, 0, 0) + [\beta] \times [\beta e^{s'}]^2.$$

Let $\mathcal{I}$ be a finite subset of $\Pi(x)$ defined as follows:

$$(8.4) \quad \mathcal{I} := \left\{ \exp(v_1 \mathfrak{d} \mathbf{v}_1 + w_1 \mathfrak{d} \mathbf{w}_1) u(r \mathfrak{d}') x : r, v_1, w_1 \in [10e^{2s'}] \cap \mathbb{Z} \right\},$$

where $\mathfrak{d} = 0.1\beta e^{-s'}$ and $\mathfrak{d}' = 0.1\beta$. For $z \in \mathcal{I}$, let $\Theta'(z) := \Theta(z) \cap \Pi(x)$.

For every $z \in \mathcal{I}$, let $\mathcal{I}_z$ be a finite subset of $\Theta'(z)$ defined as follows:

$$(8.5) \quad \mathcal{I}_z := \left\{ \exp(v_2 \mathfrak{d}'' \mathbf{v}_2 + w_2 \mathfrak{d}'' \mathbf{w}_2) z : v_2, w_2 \in [10e^{2s'}] \cap \mathbb{Z} \right\},$$

where $\mathfrak{d}'' = 0.1\beta e^{-3s'}$. For $y \in \mathcal{I}_z$, let $\Omega'(y) := \Omega(y) \cap \Theta'(z)$. Let us denote

$$\mathfrak{J} := \bigcup_{z \in \mathcal{I}} \mathcal{I}_z.$$

For any $y \in \Pi(x)$, let us define

$$(8.6) \quad \mathfrak{l}(y) := \left\{ r \in [2\beta e^{2s'}] : u(r)y \in \Pi(x) \right\}.$$

For each $z \in \mathcal{I}$ and $y \in \mathcal{I}_z$, let us write

$$y = \exp(\mathbf{w}_y^-) \exp(\mathbf{w}_y^+) u(r_y) x,$$

where

$$\begin{aligned} \mathbf{w}_y^- &= v_{2,y} \mathbf{v}_2 + w_{2,y} \mathbf{w}_2 \in B_{\mathbf{r}^-}(\beta e^{-s'}), \\ \mathbf{w}_y^+ &= v_{1,y} \mathbf{v}_1 + w_{1,y} \mathbf{w}_1 \in B_{\mathbf{r}^+}(\beta e^{s'}), \end{aligned}$$

and $r_y \in [\beta e^{2s'}]$. Let us define the curve $\mathcal{L}(y)$ as follows:

$$\mathcal{L}(y) := \{(f_y(r), f_{v,y}(r), f_{w,y}(r)) = \mathbb{P}(u(r)y) : r \in \mathcal{I}(y)\}.$$

It is straightforward to check that

$$(8.7) \quad f_{v,y}(r) = \frac{2rv_{2,y}}{2 + rv_{2,y}w_{2,y}} + v_{1,y},$$

$$(8.8) \quad f_{w,y}(r) = \frac{-2rw_{2,y}}{2 - rv_{2,y}w_{2,y}} + w_{1,y},$$

and

$$(8.9) \quad \begin{aligned} f_y(r) &= r_y + \frac{2r + 2rw_{1,y}v_{2,y}}{2 + rv_{2,y}w_{2,y}} + \frac{w_{1,y}v_{1,y} - f_{v,y}(r)f_{w,y}(r)}{2} \\ &= r_y + r + \frac{rw_{2,y}v_{1,y}}{2 - rv_{2,y}w_{2,y}} + \frac{rw_{1,y}v_{2,y}}{2 + rv_{2,y}w_{2,y}} + \frac{r(rw_{2,y}v_{2,y})^2}{4 - (rw_{2,y}v_{2,y})^2} \end{aligned}$$

From (8.9) it is easy to see that $|f_y(r) - r_y - r| \leq \beta|r|$.

Let $\mathcal{T}(y)$ denote the $2\beta e^{-s'}$ -neighborhood of $\mathcal{L}(y)$. We then assign to $\mathcal{T}(y)$ a weight

$$\mathcal{W}(y) = \tilde{\mu}(\Omega'(y)).$$

For $y_1, y_2 \in \Pi(x)$, let us define $\Delta(y_1, y_2)$ as follows: Let us write

$$y_2 = aa_0 u^*(r^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r) y_1,$$

where $aa_0 \in B^{AA_0}(2\beta)$, $r^* \in [2\beta e^{-2s'}]$, $\mathbf{w}^- = v_2 \mathbf{v}_2 + w_2 \mathbf{w}_2 \in B_{\mathbf{r}^-}(2\beta e^{-s'})$, and $\mathbf{w}^+ = v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1 \in B_{\mathbf{r}^+}(2\beta e^{s'})$. Then we define

$$(8.10) \quad \Delta(y_1, y_2) = (v_2, w_2).$$

Let us call the collection of tubes $\{\mathcal{T}(y) : y \in \mathfrak{J}\}$ with weights $\{\mathcal{W}(y)\}$ the Kakeya-type model for $(\tilde{\mu}, \Pi(x))$.

Now let us prove [Lemma 8.5](#):

Proof of Lemma 8.5. Without loss of generality, we can assume that $r = 0$. We first prove that $J_0(y) \neq \emptyset$. For any $y \in \Pi(x)$, let us write

$$y = a_y a_{0,y} u^*(r_y^*) \exp(\mathbf{w}_y^-) \exp(\mathbf{w}_y^+) u(r_y) x,$$

where $a_y a_{0,y} \in B^{AA_0}(\beta)$, $r_y^* \in [\beta e^{-2s'}]$,

$$\mathbf{w}_y^- = v_{2,y} \mathbf{v}_2 + w_{2,y} \mathbf{w}_2 \in B_{\mathbf{r}^-}(\beta e^{-s'}),$$

$$\mathbf{w}_y^+ = v_{1,y} \mathbf{v}_1 + w_{1,y} \mathbf{w}_1 \in B_{\mathbf{r}^+}(\beta e^{s'}),$$

and $r_y \in [\beta e^{2s'}]$. Then for any $r \in [\beta e^{2s'}]$,

$$u(r)y = a'_y a'_{0,y} u^*(\tilde{r}_y^*) u(\tilde{r}) \exp(\mathbf{w}_y^-) \exp(\mathbf{w}_y^+) u(r_y) x,$$

where $a'_y a'_{0,y} \in B^{AA_0}(\beta)$, $\tilde{r}_y^* \in [\beta e^{-2s'}]$, and $\tilde{r} = \tilde{r}(r)$ is a smooth function of r that satisfies the inequality $|r - \tilde{r}| \leq \beta|r|$. Moreover, by (8.9), we have

$$\begin{aligned} & u(\tilde{r}) \exp(\mathbf{w}_y^-) \exp(\mathbf{w}_y^+) u(r_y) x \\ &= a_{0,y,\tilde{r}} a_{y,\tilde{r}} u^*(r_{y,\tilde{r}}^*) \exp(\mathbf{w}_y^-(\tilde{r})) \exp(\mathbf{w}_y^+(\tilde{r})) u(f_y(\tilde{r})) x, \end{aligned}$$

where $|f_y(\tilde{r}) - r_y - \tilde{r}| \leq \beta|\tilde{r}|$. This implies that

$$|f_y(\tilde{r}) - r_y - r| \leq 2\beta|r|.$$

To prove that there exists r such that $u(r)y \in \Sigma(x)$, it suffices to show that there exists r such that $f_y(\tilde{r}) = 0$. Since $f_y(\tilde{r})$ depends on r continuously, it is enough to show that there exist r_1 and r_2 such that $f_y(\tilde{r}(r_1)) < 0$ and $f_y(\tilde{r}(r_2)) > 0$. By choosing r large (and small, respectively) enough, we can make

$$r + r_y > 2\beta|r| \text{ (and } r + r_y < -2\beta|r|, \text{ respectively),}$$

which implies that $f_y(\tilde{r}(r)) > 0$ (and $f_y(\tilde{r}(r)) < 0$, respectively). This shows that there exists r such that $u(r)y \in \Sigma(x)$.

Now fix r_0 such that

$$z = u(r_0)y = a_z a_{0,z} u^*(r_z^*) \exp(\mathbf{w}_z^-) \exp(\mathbf{w}_z^+) u(0)x.$$

We first prove that $r_0 + [\beta/2] \subset J_0(y)$; namely, for $r \in [\beta/2]$,

$$u(r)z = a_{z,r} a_{0,z,r} u^*(r_{z,r}^*) \exp(\mathbf{w}_z^-(r)) \exp(\mathbf{w}_z^+(r)) u(f_z(r)) x$$

with $|f_z(r)| \leq \beta$. In fact, the above argument shows that $|f_z(r) - r| \leq \beta|r|$. Therefore, for $|r| \leq \beta/4$, we have $|f_z(r)| \leq (1 + \beta)|r| \leq \beta$. This shows that $r_0 + [\beta/2] \subset J_0(y)$.

Similar argument shows that if $|f_z(r)| \leq \beta$, then $|r| \leq (1 - \beta)^{-1}|f_z(r)| \leq 2\beta$. This shows that $J_0(y) \subset r_0 + [4\beta]$.

This completes the proof. $\square$

We will need the following lemmata:

LEMMA 8.8. *We have*

$$\Pi(x) = \bigcup_{z \in \mathcal{I}} \Theta'(z).$$

Moreover, for any $z' \in \Pi(x)$, there are at most N_2 different $z \in \mathcal{I}$ such that

$$\Theta(z') \cap \Theta(z) \neq \emptyset.$$

Proof. We first prove that for any $z' \in \Pi(x)$, there exists $z \in \mathcal{I}$ such that $z' \in \Theta(z)$. In fact, let us write

$$z' = a_{z'} a_{0,z'} u^*(r_{z'}^*) \exp(\mathbf{w}_{z'}^-) \exp(v_{1,z'} \mathbf{v}_1 + w_{1,z'} \mathbf{w}_1) u(r_{z'}) x,$$

where $a_{z'} \in B^A(\beta)$, $a_{0,z'} \in B^{A_0}(\beta)$, $r_{z'}^* \in [\beta e^{-2s'}]$, $\mathbf{w}_{z'}^- \in B_{\mathbf{r}-}(\beta e^{-s'})$, and

$$(r_{z'}, v_{1,z'}, w_{1,z'}) \in [\beta e^{2s'}] \times [\beta e^{s'}]^2.$$

Let us choose $z \in \mathcal{I}$ such that

$$z = \exp(v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1) u(r_z) x$$

and such that $|r_z - r_{z'}| \leq 0.2\beta$ and

$$\max\{|v_{1,z} - v_1|, |w_{1,z} - w_1|\} \leq 0.2\beta e^{-s'}.$$

We claim that $z' \in \Theta(z)$. In fact, by direct calculation, we have

$$z' = a_{z'} a_{0,z'} u^*(r_{z'}^*) \exp(\mathbf{w}_{z'}^-) \exp(v'_1 \mathbf{v}_1 + w'_1 \mathbf{w}_1) u(r') z,$$

where $v'_1 = v_{1,z'} - v_1$, $w'_1 = w_{1,z'} - w_1$, and $r' = r_{z'} - r_z - \frac{v'_1 w_1 - v_1 w'_1}{2}$. Note that

$$\max\{|v'_1|, |w'_1|\} = \max\{|v_{1,z'} - v_1|, |w_{1,z'} - w_1|\} \leq 0.2\beta e^{-s'},$$

and

$$|r'| \leq |r_{z'} - r_z| + |v'_1 w_1| + |v_1 w'_1| \leq 0.1\beta + 2 \times 0.2\beta e^{-s'} \times \beta e^{s'} \leq 0.3\beta.$$

Thus, we have $z' \in \Theta(z)$.

We then prove that there are at most N_2 different $z \in \mathcal{I}$ such that

$$\Theta(z') \cap \Theta(z) \neq \emptyset.$$

In fact, let us take

$$\tilde{z} \in \Theta(z') \cap \Theta(z).$$

Then we have

$$z' \in \Theta_{s'}^{-1} \tilde{z} \subset \Theta_{s'}^{-1} \Theta_{s'} z.$$

By [Lemma 8.2](#), we have

$$z' \in \bar{\Theta}_{s'} z;$$

namely,

$$z' = \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}) z,$$

where $\bar{\mathbf{w}} \in B_{\mathbf{r}_1 + \mathbf{r}_2}(2.21\beta e^{-s'})$ and $\bar{\mathbf{h}} \in \mathfrak{q}^{A_0 H}(2s', 2.21\beta)$. By rearranging the equation above, we get

$$z' = \exp(b\mathbf{a} + b_0\mathbf{a}_0 + r^*\mathbf{u}^*) \exp(\mathbf{w}'') u(r) z,$$

where $b, b_0, r \in [2.22\beta]$, $r^* \in [2.22\beta e^{-2s'}]$, and $\mathbf{w}'' \in B_{\mathbf{r}_1 + \mathbf{r}_2}(2.22\beta e^{-s'})$. By [Lemma 8.4](#), we have

$$\exp(\mathbf{w}'') = \exp(\tilde{b}\mathbf{a} + \tilde{b}_0\mathbf{a}_0 + \tilde{r}^*\mathbf{u}^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(\tilde{r}),$$

where $\mathbf{w}^- \in B_{\mathbf{r}-}(2.3\beta e^{-s'})$, $\mathbf{w}^+ \in B_{\mathbf{r}+}(2.3\beta e^{-s'})$, $\tilde{b}, \tilde{b}_0, \tilde{r}^*, \tilde{r} \in [0.1\beta e^{-2s'}]$.

Thus,

$$(8.11) \quad z' = a_0 a u^*(\tilde{r}^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(\tilde{r}) z,$$

where $a_0a \in B^{A_0A}(3.1\beta)$, $\bar{r}^* \in [3.1\beta e^{-2s'}]$, $\bar{r} \in [3.1\beta]$, $\mathbf{w}^- \in B_{\mathbf{r}^-}(2.3\beta e^{-s'})$, and $\mathbf{w}^+ \in B_{\mathbf{r}^+}(2.3\beta e^{-s'})$. Let us consider $\mathbb{P}(z)$ and $\mathbb{P}(z')$. From (8.11), we have that

$$\mathbb{P}(z) \in \mathbb{P}(z') + [3.1\beta] \times [2.3\beta e^{-s'}]^2.$$

From the definition of $\mathcal{I}$, we have that there are at most N_2 different $z \in \mathcal{I}$ satisfying the above condition. $\square$

LEMMA 8.9. *For any $z \in \mathcal{I}$,*

$$\Theta'(z) = \bigcup_{y \in \mathcal{I}_z} \Omega'(y).$$

Moreover, for any $y' \in \Theta(z)$, there are at most N_2 different $y \in \mathcal{I}_z$ such that

$$\Omega(y) \cap \Omega(y') \neq \emptyset.$$

Before proving Lemma 8.9, let us introduce a coordinate system in $\Theta(z)$:

Definition 8.10. Given $z \in \mathcal{I}$ and $y \in \Theta'(z)$, let us define the coordinate of y in $\Theta'(z)$, denoted by $\mathfrak{D}_z(y) \in \mathbb{R}^2$ as follows: Let us write

$$y = a_0au(r_y) \exp(\mathbf{w}^+) \exp(\mathbf{w}^-) u^*(r_y^*) z,$$

where $a_0a \in A_0A$, $\mathbf{w}^+ \in \mathbf{r}^+$ and $\mathbf{w}^- = v_2\mathbf{v}_2 + w_2\mathbf{w}_2 \in \mathbf{r}^-$. Then we define

$$\mathfrak{D}_z(y) := (v_2, w_2).$$

Let us prove Lemma 8.9:

Proof of Lemma 8.9. We first prove that for any $z' \in \Theta'(z)$, there exists $y \in \mathcal{I}_z$ such that $z' \in \Omega(y)$. By Lemma 8.1, $\Theta_{s'} \subset \hat{\Theta}_{s'}$. Therefore, we can write

$$z' = a_0au(r) \exp(\mathbf{w}^+ + \mathbf{w}^-) u^*(r^*) z,$$

where $a_0a \in B^{A_0A}(1.1\beta)$, $r \in [1.1\beta]$, $r^* \in [1.1\beta e^{-2s'}]$, $\mathbf{w}^+ \in B_{\mathbf{r}^+}(1.1\beta e^{-s'})$ and $\mathbf{w}^- = v_2\mathbf{v}_2 + w_2\mathbf{w}_2 \in B_{\mathbf{r}^-}(1.1\beta e^{-s'})$.

Let $y \in \mathcal{I}_z$ be a point such that

$$y = \exp(\bar{\mathbf{w}}^-) z,$$

with $\|\bar{\mathbf{w}}^- - \mathbf{w}^-\| \leq 0.2\beta e^{-3s'}$.

We will prove that $z' \in \Omega(y)$. In fact, we have

$$(8.12) \quad z' = a_0au(r) \exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(-\bar{\mathbf{w}}^-) u^*(r^*) y.$$

By Lemma 2.3, we have that

$$\exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(-\bar{\mathbf{w}}^-) = \exp(\mathbf{w}^+ + \mathbf{w}^- - \bar{\mathbf{w}}^- + \tilde{\mathbf{w}}) \exp(\tilde{\mathbf{h}}),$$

where $\tilde{\mathbf{w}} \in B_{\mathbf{r}_1 + \mathbf{r}_2}(0.1\beta e^{-3s'})$, and $\tilde{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathfrak{h}}(0.1\beta e^{-2s'})$. Let us denote

$$\mathbf{w} = \mathbf{w}^+ + \mathbf{w}^- - \bar{\mathbf{w}}^- + \tilde{\mathbf{w}} \in \mathbf{r}_1 + \mathbf{r}_2,$$

and let us write

$$\exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(-\bar{\mathbf{w}}^-) = \exp(\mathbf{w}) \exp(\tilde{\mathbf{h}}).$$

Then the above estimates imply that $\|p_-(\mathbf{w})\| \leq \beta e^{-3s'}$. By plugging the above equation into (8.12), we get that $z' \in \Omega(y)$.

Let us prove that there are at most N_2 different $y \in \mathcal{I}_z$ such that

$$\Omega(z') \cap \Omega(y) \neq \emptyset.$$

Let us take

$$y' \in \Omega(z') \cap \Omega(y).$$

Then

$$y' = \exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(\mathbf{h})y = \exp(\tilde{\mathbf{w}}^+ + \tilde{\mathbf{w}}^-) \exp(\mathbf{h}')z',$$

where

$$\begin{aligned} \mathbf{w}^+, \tilde{\mathbf{w}}^+ &\in B_{\mathbf{r}^+}(2\beta e^{-s'}), \\ \mathbf{w}^-, \tilde{\mathbf{w}}^- &\in B_{\mathbf{r}^-}(2\beta e^{-3s'}), \end{aligned}$$

and

$$\mathbf{h}, \mathbf{h}' \in \mathfrak{q}^{A_0 H}(2s', 2\beta).$$

Then $z' = \exp(-\mathbf{h}') \exp(-\tilde{\mathbf{w}}^+ - \tilde{\mathbf{w}}^-) \exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(\mathbf{h})y$.

By Lemma 2.3,

$$\exp(-\tilde{\mathbf{w}}^+ - \tilde{\mathbf{w}}^-) \exp(\mathbf{w}^+ + \mathbf{w}^-) = \exp(\tilde{\mathbf{w}}^+ + \tilde{\mathbf{w}}^-) \exp(\tilde{\mathbf{h}}),$$

where $\tilde{\mathbf{w}}^+ \in B_{\mathbf{r}^+}(4.1\beta e^{-s'})$, $\tilde{\mathbf{w}}^- \in B_{\mathbf{r}^-}(4.1\beta e^{-3s'})$, $\tilde{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathfrak{h}}(0.1\beta e^{-2s'})$.

Thus,

$$z' = \exp(-\mathbf{h}') \exp(\tilde{\mathbf{w}}^+ + \tilde{\mathbf{w}}^-) \exp(\tilde{\mathbf{h}}) \exp(\mathbf{h})y.$$

By rearranging the above equation, we have

$$z' = a_0 a u(r) \exp(\mathbf{v}^+ + \mathbf{v}^-) u^*(r^*)y,$$

where $a_0 a \in B^{A_0 A}(2.3\beta)$, $r \in [2.3\beta]$, $r^* \in [2.3\beta e^{-2s'}]$, $\mathbf{v}^+ \in B_{\mathbf{r}^+}(4.3\beta e^{-s'})$, and $\mathbf{v}^- \in B_{\mathbf{r}^-}(4.3\beta e^{-3s'})$. By Lemma 8.4,

$$\exp(\mathbf{v}^+ + \mathbf{v}^-) = \tilde{a}_0 \tilde{a} u(\tilde{r}) \exp(\tilde{\mathbf{v}}^+) \exp(\tilde{\mathbf{v}}^-) u^*(\tilde{r}^*),$$

where $\tilde{a}_0 \tilde{a} \in B^{A_0 A}(0.1\beta e^{-2s'})$, $\tilde{r}, \tilde{r}^* \in [0.1\beta e^{-2s'}]$, $\tilde{\mathbf{v}}^+ \in B_{\mathbf{r}^+}(4.4\beta e^{-s'})$, and $\tilde{\mathbf{v}}^- \in B_{\mathbf{r}^-}(4.4\beta e^{-3s'})$. This implies that

$$z' = a_0 a u(r) \tilde{a}_0 \tilde{a} u(\tilde{r}) \exp(\tilde{\mathbf{v}}^+) \exp(\tilde{\mathbf{v}}^-) u^*(\tilde{r}^*) u^*(r^*)y.$$

Considering the coordinates of z' and y in $\Theta(z)$, we have

$$\|\mathfrak{D}_z(z') - \mathfrak{D}_z(y)\| = \|\mathbf{v}^-\| \leq 4.4\beta e^{-3s'}.$$

This shows that there are at most N_2 different $y \in \mathcal{I}_z$ satisfying this condition.

This completes the proof. $\square$

Remark 8.11. Let us denote

$$(8.13) \quad \mathfrak{C} := \left[1.2\beta e^{-s'}\right]^2.$$

From [Lemma 8.9](#), we can see that for any $z' \in \Theta(z)$,

$$\mathfrak{D}_z(z') \in \mathfrak{C}.$$

LEMMA 8.12. *Suppose*

$$y_2 = \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_0) y_1,$$

where $|r_0| \leq \beta e^{2s'}$, $\mathbf{w}^- = v_2 \mathbf{v}_2 + w_2 \mathbf{w}_2 \in B_{\mathbf{r}^-}(\beta e^{-s'})$, and $\mathbf{w}^+ = v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1 \in B_{\mathbf{r}^+}(\beta e^{s'})$. For $r \in [\beta e^{2s'}]$, let us write $u(r)y_2$ as

$$u(r)y_2 = aa_0 u^* \exp(\mathbf{w}^-(r)) \exp(\mathbf{w}^+(r)) u(f(r)) y_1,$$

where $a \in A$, $a_0 \in A_0$, $u^* \in U^*$, $\mathbf{w}^+(r) \in \mathbf{r}^+$, and

$$\mathbf{w}^-(r) = v_2(r) \mathbf{v}_2 + w_2(r) \mathbf{w}_2 \in \mathbf{r}^-.$$

Then

$$(8.14) \quad v_2(r) = \frac{2v_2}{2 - rw_2v_2},$$

and

$$(8.15) \quad w_2(r) = \frac{2w_2}{2 + rw_2v_2}.$$

In particular, for any $r' \in [\beta e^{2s'}]$, if we denote $y'_1 = u(r')y_1$ and $y'_2 = u(r)y_2$, we have

$$\|\Delta(y'_1, y'_2)\| \asymp \|\Delta(y_1, y_2)\|.$$

Proof. This can be verified by direct calculation. $\square$

The following lemma tells that $\Omega(y)$ is invariant along the U -orbits, modulo necessary time change:

LEMMA 8.13. *Let $\tilde{\mu}$ and $\Pi(x)$ be as in [Definition 8.7](#). Let us denote*

$$(8.16) \quad \hat{\Omega}_{s'} := Q_{3s'}^{s'}(2.1\beta) Q^{A_0H}(2s', 2.1\beta).$$

For $y \in \mathfrak{J}$, let us denote

$$(8.17) \quad U\Omega'(y) := \{u(r')y' : y' \in \Omega'(y), r' \in \mathfrak{l}(y')\}.$$

For $r \in \mathfrak{l}(y)$, let us denote

$$(8.18) \quad \Omega(y, r) := U\Omega'(y) \cap \Sigma^{\tilde{r}},$$

where $\tilde{r}$ satisfies the following: $u(r)y \in \Sigma^{\tilde{r}}$. Then

$$\Omega(y, r) \subset \hat{\Omega}(u(r)y) := \hat{\Omega}_{s'} u(r)y.$$

In particular, we have

$$\tilde{\mu}(\Omega'(y)) \ll \tilde{\mu}(\hat{\Omega}(u(r)y)).$$

Proof. For $y' \in \Omega'(y)$, we have

$$y = \exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(\mathbf{h}) y',$$

where $\mathbf{w}^+ \in B_{\mathbf{r}^+}(2.01\beta e^{-s'})$, $\mathbf{w}^- \in B_{\mathbf{r}^-}(2.01\beta e^{-3s'})$, and $\mathbf{h} \in \mathfrak{q}^{A_0H}(2s', 2\beta)$. Then

$$\begin{aligned} u(r)y &= u(r) \exp(\mathbf{w}^+ + \mathbf{w}^-) \exp(\mathbf{h}) y' \\ &= \exp(\text{Ad}_{u(r)}(\mathbf{w}^+ + \mathbf{w}^-)) u(r) \exp(\mathbf{h}) y' \\ &= \exp(\mathbf{w}^+ + \tilde{\mathbf{w}}^+ + \mathbf{w}^-) \exp(\tilde{\mathbf{h}}) u(r') y', \end{aligned}$$

where $\tilde{\mathbf{w}}^+ \in \mathbf{r}^+$ with $\|\tilde{\mathbf{w}}^+\| = |r|\|\mathbf{w}^-\|$, and $\tilde{\mathbf{h}} \in \mathfrak{q}^{A_0H}(2s', 2.01\beta)$. Note that $|r| \leq \beta e^{2s'}$ and $\|\mathbf{w}^-\| \leq 2.01\beta e^{-3s'}$. We have $\|\tilde{\mathbf{w}}^+\| \leq 0.01\beta e^{-s'}$. This implies that

$$u(r')y' \in \hat{\Omega}(u(r)y).$$

Note also that $u(r')y'$ and $u(r)y$ are in the same cross-section $\Sigma^{\tilde{r}}$. By [Lemma 8.5](#), $u(\mathbf{l}(y'))y' \cap \Sigma^{\tilde{r}}$ is contained in an interval of length $\asymp \beta$. This implies that for any $y' \in \Omega(y)$,

$$u(\mathbf{l}(y'))y' \cap \Sigma^{\tilde{r}} \subset \hat{\Omega}(u(r)y).$$

Thus, we have

$$\Omega(y, r) \subset \hat{\Omega}(u(r)y).$$

Note that $\Omega'(y)$ and $\Omega(y, r)$ are both cross-sections of $U\Omega'(y)$ along U -direction. Thus, we have

$$\tilde{\mu}(\Omega(y, r)) \asymp \tilde{\mu}(\Omega'(y)) \asymp e^{-2s'} \tilde{\mu}(U\Omega'(y)).$$

This implies that

$$\tilde{\mu}(\Omega(y)) \ll \tilde{\mu}(\hat{\Omega}(u(r)y)). \quad \square$$

LEMMA 8.14. *Let s' be a positive number, and let $\Pi(x)$ be as in [Definition 8.7](#). Let us fix a cross-section $\Sigma^r \subset \Pi(x)$ defined as above. Then for any $y_1, y_2 \in \mathfrak{J} \cap \Sigma^r$, suppose there exist L_1, L_2 and $z \in \mathcal{I}$ such that*

$$\Omega(y_i, L_i) \cap \Theta'(z) \neq \emptyset, \text{ for } i = 1, 2,$$

where $\Omega(y_i, L_i)$ is defined as in [\(8.18\)](#).

Let us denote $\tilde{y}_i = u(L_i)y_i$ and write

$$\tilde{y}_1 = a_0 a u^*(r^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_0) \tilde{y}_2,$$

where $a_0a \in A_0A$, $\mathbf{w}^- = v_2\mathbf{v}_2 + w_2\mathbf{w}_2 \in \mathbf{r}^-$, and $\mathbf{w}^+ = v_1\mathbf{v}_1 + w_1\mathbf{w}_1 \in \mathbf{r}^+$. Then $\mathbf{w}^+ \in B_{\mathbf{r}^+}(4.6\beta e^{-s'})$. Moreover, let

$$(8.19) \quad I(y_1, y_2) := \{r_1 \in \mathcal{I}(y_1) : \exists r_2, z' \in \mathcal{I}, \text{ s.t. } \Omega(y_i, r_i) \cap \Theta'(z') \neq \emptyset, i = 1, 2\}.$$

Then there exists an interval I , such that $I(y_1, y_2) \subset I$ with

$$|I| \asymp \min \left\{ \beta e^{2s'}, \beta e^{-s'} \|\mathbf{w}^-\|^{-1} \right\}.$$

Proof. By Lemma 8.13, we have $\Omega(y_i, L_i) \subset \hat{\Omega}(\tilde{y}_i)$. Therefore, we have $\hat{\Omega}(\tilde{y}_1) \cap \Theta(z) \neq \emptyset$, which implies that

$$\tilde{y}_1 \in \hat{\Omega}_{s'}^{-1} \Theta_{s'} z \subset \Theta_{s'}^{-1} \Theta_{s'} z.$$

Similarly, we have $\hat{\Omega}(\tilde{y}_2) \cap \Theta(z) \neq \emptyset$ that implies that

$$z \in \Theta_{s'}^{-1} \Omega_{s'} \tilde{y}_2 \subset \Theta_{s'}^{-1} \Theta_{s'} \tilde{y}_2.$$

Therefore,

$$\tilde{y}_1 \in \Theta_{s'}^{-1} \Theta_{s'} \Theta_{s'}^{-1} \Theta_{s'} \tilde{y}_2.$$

By Corollary 8.3,

$$\tilde{y}_1 \in \tilde{\Theta}_{s'} \tilde{y}_2.$$

Then we have

$$\tilde{y}_1 = \tilde{a}_0 \tilde{a} u^*(\tilde{r}^*) \exp(\tilde{\mathbf{w}}^+ + \tilde{\mathbf{w}}^-) u(\tilde{r}) \tilde{y}_2,$$

where $\mathbf{w}^+ \in B_{\mathbf{r}^+}(4.5\beta e^{-s'})$. By Lemma 8.4, if we write

$$\tilde{y}_1 = a_0 a u^*(r^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_0) \tilde{y}_2,$$

we will have $\|\mathbf{w}^+ - \tilde{\mathbf{w}}^+\| \leq 0.01\beta e^{-3s'}$. This implies that $\mathbf{w}^+ \in B_{\mathbf{r}^+}(4.6\beta e^{-s'})$.

For any $r_1 \in [2\beta e^{2s'}]$, we have that

$$\begin{aligned} u(r_1) \tilde{y}_1 &= u(r_1) a a_0 u^*(r^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_0) \tilde{y}_2 \\ &= a' a'_0 u_1^* u(r'_1) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_0) \tilde{y}_2, \end{aligned}$$

where $a' a'_0 \in B^{A_0 A}(\beta)$, $u_1^* \in B^{U^*}(\beta e^{-2s'})$, and $|r_1 - r'_1| \leq \beta |r_1|$. By (8.7), (8.8), and (8.9), we have that

$$u(r_1) \tilde{y}_1 = \bar{a} \bar{a}_0 \bar{u}^* \exp(\mathbf{w}^-(r'_1)) \exp(\mathbf{w}^+(r'_1)) u(f(r'_1)) \tilde{y}_2$$

with $\mathbf{w}^+(r'_1) = f_v(r'_1)\mathbf{v}_1 + f_w(r'_1)\mathbf{w}_1$, where

$$f_v(r'_1) = v_1 + \frac{2v_2 r'_1}{2 + r'_1 v_2 w_2},$$

and

$$f_w(r'_1) = w_1 + \frac{-2w_2 r'_1}{2 - r'_1 v_2 w_2}.$$

By our previous argument, we can see that $r_1 \in I(y_1, y_2)$ implies that $\|\mathbf{w}^+(r'_1)\| \leq 4.6\beta e^{-s'}$.

Note that

$$\left| \frac{2v_2r'_1}{2+r'_1v_2w_2} - v_2r'_1 \right| \leq \beta|v_2r'_1|.$$

If $|f_v(r'_1)| \leq 4.6\beta e^{-s'}$, then $|v_2r'_1| \leq 10\beta e^{-s'}$ implying that $|r'_1| \leq 10\beta e^{-s'}|v_2|^{-1}$.

By applying the same argument to $f_w(r'_1)$, we have that $|f_w(r'_1)| \leq 4.6\beta e^{-s'}$ implies that $|r'_1| \leq 10\beta e^{-s'}|w_2|^{-1}$.

This completes the proof. $\square$

Recall that in [Definition 8.10](#), we define a coordinate system in $\Theta'(z)$ for any $z \in \mathcal{I}$. By [Remark 8.11](#), for any $z' \in \Theta'(z)$, its coordinate $\mathfrak{D}_z(z')$ is contained in

$$\mathfrak{C} = [1.2\beta e^{-s'}]^2.$$

The following lemma is crucial for us to study the structure of the Keakeya-type model:

LEMMA 8.15. *Recall that $I(y_1, y_2)$ is defined as in (8.19). Let us fix a cross-section Σ^r . Let $z \in \mathcal{I} \cap \Sigma^r, y \in \mathfrak{J} \cap \Sigma^r$. Then the set of all coordinates $\mathfrak{D}_z(\bar{z})$ of the elements $\bar{z} \in \mathcal{I}_z$ such that $I(\bar{z}, y) \neq \emptyset$ and such that there is some $r \in I(\bar{z}, y)$ with $|r| \geq \beta^{100}e^{2s'}$ is contained in the $\beta^{-100}e^{-3s'}$ neighborhood of the hyperbola*

$$(8.20) \quad \left\{ (X, Y) \in \mathfrak{C} : (X - X_0)(Y - Y_0) + \frac{X - X_0}{v_1} + \frac{Y - Y_0}{w_1} = 0 \right\},$$

where v_1, w_1, X_0, Y_0 are constants depending on y, z with bounds $(X_0, Y_0) \in \mathfrak{C}$ and $(v_1, w_1) \in [\beta e^{s'}]^2$. The set of such coordinates $\mathfrak{D}_z(\bar{z})$ is also contained in the box

$$(8.21) \quad \mathfrak{B}(\mathcal{T}(y), z) := (X_0, Y_0) + [\beta^{-100}e^{-2s'}\|(v_1, w_1)\|]^2.$$

Moreover, if there exists $\bar{z} \in \mathcal{I}_z$ such that there is some $r \in I(\bar{z}, y)$ with $|r| \geq \beta^{100}e^{2s'}$ and

$$\|\Delta(\bar{z}, y)\| \leq \beta e^{-s'-j},$$

then

$$\|(v_1, w_1)\| \leq \beta^2 e^{s'-j},$$

and the size of $\mathfrak{B}(\mathcal{T}(y), z)$ does not exceed $\beta^{-98}e^{s'-j}$.

Proof. There exist $y' \in \Omega(y)$ and $z' \in \Theta(z)$ such that

$$z' = \exp(\mathbf{w}^+)u(r_{y'})y',$$

where $\mathbf{w}^+ = v_1\mathbf{v}_1 + w_1\mathbf{w}_1$, and

$$z' = \exp(\mathbf{w}_0^-)z,$$

where $\mathbf{w}_0^- = X_0\mathbf{v}_2 + Y_0\mathbf{w}_2$. We can also make sure that

$$\min\{|v_1|, |w_1|\} \geq \beta e^{-s'}.$$

Without loss of generality, let us assume $|w_1| \geq |v_1|$. The opposite case can be handled similarly.

Suppose $\bar{z} = \exp(\mathbf{w}^-)z'$ (where $\mathbf{w}^- = v_2\mathbf{v}_2 + w_2\mathbf{w}_2$) satisfies the following: $I(\bar{z}, y) \neq \emptyset$. Then by (8.7), (8.8), and (8.9), we have that

$$u(r)\bar{z} = a_{0,r}a_ru^*(f^*(r)) \exp(\mathbf{f}_-(r)) \exp(\mathbf{f}_+(r))u(f(r))y',$$

where $\mathbf{f}_+(r) = f_v(r)\mathbf{v}_1 + f_w(r)\mathbf{w}_1$,

$$f_v(r) = \frac{2rv_2}{2 + rv_2w_2} + v_1,$$

and

$$f_w(r) = \frac{-2rw_2}{2 - rv_2w_2} + w_1.$$

By Lemma 8.14, $r \in I(\bar{z}, y)$ implies that $|f_v(r)|, |f_w(r)| \leq 4.6\beta e^{-s'}$. Let us denote $\Delta = v_2w_2/2$. Then we have

$$(8.22) \quad r^{-1} + \Delta = -\frac{v_2}{v_1} \left(1 + O(\beta e^{-s'} v_1^{-1})\right),$$

and

$$(8.23) \quad r^{-1} - \Delta = \frac{w_2}{w_1} \left(1 + O(\beta e^{-s'} w_1^{-1})\right).$$

Note that $|v_1| \geq \beta e^{-s'}$. So we have

$$|\beta e^{-s'} v_1^{-1}| \leq 1.$$

Note that

$$(8.24) \quad |r^{-1}| \geq \beta^{-1} e^{-2s'} \geq \beta^2 e^{-2s'} \geq |\Delta|.$$

Since $|r| \geq \beta^{100} e^{2s'}$, we get

$$(8.25) \quad |v_2/v_1|, |w_2/w_1| \leq \beta^{-100} e^{-2s'}.$$

This implies that

$$\|(v_2, w_2)\| \leq \beta^{-100} e^{-2s'} \|(v_1, w_1)\|.$$

Moreover, (8.23) and (8.22) imply that

$$v_2w_2 + v_1^{-1}v_2 + w_1^{-1}w_2 = O(\beta^{-99} e^{-3s'} v_1^{-1}) + O(\beta^{-99} e^{-3s'} w_1^{-1}).$$

This implies that the coordinate of $\bar{z}$, $\mathfrak{D}_z(\bar{z}) = (v_2 + X_0, w_2 + Y_0)$ is contained in $\beta^{-100} e^{-3s'}$ neighborhood of the hyperbola

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0,$$

and it is also contained in $(X_0, Y_0) + \left[\beta^{-100} e^{-2s'} \|(v_1, w_1)\|\right]^2$.

Now let us assume that $\|\Delta(\bar{z}, y)\| \leq e^{-s'-j}$. Then we have

$$\bar{z} = \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_{y'}) y',$$

where $\mathbf{w}^- = v_2 \mathbf{v}_2 + w_2 \mathbf{w}_2 \in B_{\mathbf{r}^-}(e^{-s'-j})$. Then by (8.22), (8.23), and (8.24), we have that

$$|v_2/v_1|, |w_2/w_1| \geq \beta^{-1} e^{-2s'}.$$

This implies that

$$\|(v_1, w_1)\| \leq \beta e^{2s'} \|(v_2, w_2)\| \leq \beta^2 e^{s'-j}.$$

By (8.21), we have that the size of $\mathfrak{B}(\mathcal{T}(y), z)$ does not exceed $\beta^{-98} e^{s'-j}$. This completes the proof. $\square$

Remark 8.16.

- (1) Note that $I(y_1, y_2)$ will change if we replace y_1 with $y'_1 \in \Omega(y_1)$. However, there exists a one-to-one map

$$P_{y_1, y'_1} : I(y_1, y_2) \rightarrow I(y'_1, y_2)$$

such that $|P_{y_1, y'_1}(r') - r'| \leq \beta |r'|$ for every $r' \in I(y_1, y_2)$.

- (2) If $|v_1/w_1| \leq e^{-s'}$ or $|w_1/v_1| \leq e^{-s'}$, then the neighborhood described in (8.20) agrees with the $\beta^{-100} e^{-3s'}$ neighborhood of the straight line

$$v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0.$$

Let us denote the $\beta^{-100} e^{-3s'}$ -neighborhood of the curve (8.20) by

$$\mathcal{C}(\mathcal{T}(y), z) \subset \mathfrak{C}.$$

By misuse of notation, we will write $\Omega(\bar{z}) \subset \mathcal{C}(\mathcal{T}(y), z)$ when we actually mean that $\mathfrak{D}_z(\bar{z}) \in \mathcal{C}(\mathcal{T}(y), z)$. Also, note that if $\mathfrak{D}_z(\bar{z}) \in \mathcal{C}(\mathcal{T}(y), z)$ but

$$\mathfrak{D}_z(\bar{z}) \notin \mathfrak{B}(\mathcal{T}(y), z),$$

then there does not exist $r' \in I(\bar{z}, y)$ with $r' \geq \beta^{100} e^{2s'}$.

9. Dimension Improvement — unstructured part

Recall that our goal is to prove the following proposition:

PROPOSITION 9.1. *Suppose that $\mathcal{F}_i$ is (d_i, s_i) -good. Let $\mu = a(s_i)_* \mu_{\mathcal{F}_i}$. Let us fix a full playground $\Pi(x)$ and denote $\tilde{\mu} = \mu_{\Pi(x)}$. Then $\tilde{\mu}$ admits the following decomposition:*

$$\tilde{\mu} = \nu_{\mathcal{R}_i} \tilde{\mu}_{\mathcal{R}_i} + \nu_{i+1} \tilde{\mu}_{i+1}$$

such that $\nu_{\mathcal{R}_i} \leq \beta^{20}$, and μ_{i+1} is $(d_i + \varepsilon_1, s_{i+1})$ -good. Recall that $s_{i+1} = s_i/2$.

The proof of this proposition occupies the rest of this section as well as [Sections 10](#) and [11](#).

Let us define the heavy part of $\Pi(x)$:

Definition 9.2. For $z \in \mathcal{I}$, we call $\Theta'(z)$ a *heavy neighborhood* if

$$\tilde{\mu}(\Theta'(z)) > e^{-2s'} e^{-(d_i + \varepsilon_1)s'},$$

and we call $\Theta'(z)$ *extremely heavy* if

$$\tilde{\mu}(\Theta'(z)) > e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'}.$$

Definition 9.3. Let $\mathcal{R}_i \subset \Pi(x)$ denote the union of heavy neighborhoods. Then $\mathcal{R}_i$ will be said to be the *heavy part* of $\Pi(x)$.

To prove [Proposition 9.1](#), it suffices to show that

$$(9.1) \quad \tilde{\mu}(\mathcal{R}_i) \leq \beta^{20}.$$

In fact, if this is the case, we can put $\tilde{\mu}_{\mathcal{R}_i} = \tilde{\mu}^{-1}(\mathcal{R}_i)\tilde{\mu}|_{\mathcal{R}_i}$, and $\tilde{\mu}_{i+1} = \tilde{\mu}^{-1}(\Pi(x) \setminus \mathcal{R}_i)\tilde{\mu}|_{\Pi(x) \setminus \mathcal{R}_i}$. Then, since $\Pi(x) \setminus \mathcal{R}_i$ does not have any heavy neighborhood, we have, for any $z \in \mathcal{I}$,

$$\tilde{\mu}_{i+1}(\Theta'(z)) \leq e^{-2s'} e^{-(d_i + \varepsilon_1)s'}.$$

Since every neighborhood of the form $Q_{s'}(\beta)B^{A_0H}(\beta)y$ contains at most $e^{2s'}$ different $\Theta'(z)$'s, we conclude that $\tilde{\mu}_{i+1}$ is $(d_i + \varepsilon_1, s')$ -good, which is what we wanted.

For the rest of the proof, we will focus on proving [\(9.1\)](#).

We first prove the following lemma:

LEMMA 9.4. *Let us denote $s' = s_{i+1}$. For any $\Omega(y) = \Omega_{s'}y$, we have*

$$(9.2) \quad \tilde{\mu}(\Omega'(y)) \leq \beta^{-15} e^{-2s'} e^{-2d_i s'}.$$

Proof. First note that

$$\mu(\Omega(y)) = \mu_{\mathcal{F}_i}(a(-2s')\Omega(y)),$$

and

$$a(-2s')\Omega(y) = Q_{2s'}(2\beta)R^{A_0H}(2s', 2\beta)y',$$

where $y' = a(-2s')y$. Noting that $\mu_{\mathcal{F}_i}$ is a normalized measure on U -orbits, we have

$$\mu_{\mathcal{F}_i}(Q_{2s'}(2\beta)R^{A_0H}(2s', 2\beta)y') \leq e^{-2s'} \mu_{\mathcal{F}_i}(Q_{2s'}(2\beta)B^{A_0H}(2\beta)y').$$

By our hypothesis that $\mu_{\mathcal{F}_i}$ is $(d_i, 2s')$ -good, we have

$$\mu_{\mathcal{F}_i}(Q_{2s'}(2\beta)B^{A_0H}(2\beta)y') \leq e^{-2d_i s'},$$

which implies the claim.

Since $\Pi(x)$ is a full playground, we have that for any $y \in \mathfrak{J}$, [\(9.2\)](#) holds. $\square$

Let us further decompose $\mathcal{R}_i$ as follows:

Definition 9.5. For integer $j \geq 1$, let us denote

$$\Xi_j(z') := \Xi_{s',j} z' \cap \Theta'(z'),$$

where

$$\Xi_{s',j} := Q_{s'+j}^{s'} Q^{A_0 H}(2s', 1.1\beta).$$

Given a heavy neighborhood $\Theta'(z)$, we call $\Theta'(z)$ *strange* if there exists $\Xi_{C_1 \varepsilon_1 s'}(z') \subset \Theta'(z)$ such that

$$\tilde{\mu}_\tau(\Xi_{C_1 \varepsilon_1 s'}(z')) \geq (1 - \beta) \tilde{\mu}_\tau(\Theta'(z)).$$

Otherwise, we call it a *normal neighborhood*. Let $\bar{\mathcal{R}}_i \subset \mathcal{R}_i$ denote the union of all the heavy neighborhoods that are also normal. Then we have the following decomposition:

$$(9.3) \quad \mathcal{R}_i = \bar{\mathcal{R}}_i \cup \bigcup_{j=C_1 \varepsilon_1 s'}^{2s'} \mathcal{R}_{i,j},$$

where $\mathcal{R}_{i,j}$ denotes the union of heavy neighborhoods $\Theta'(z)$ such that there exists $\Xi_j(z') \subset \Theta'(z)$ satisfying

$$(9.4) \quad \tilde{\mu}_\tau(\Xi_j(z')) \geq (1 - \beta)^j \tilde{\mu}_\tau(\Theta'(z)),$$

and

$$(9.5) \quad \tilde{\mu}_\tau(\Xi_{j+1}(z'')) < (1 - \beta) \tilde{\mu}_\tau(\Xi_j(z'))$$

for any $\Xi_{j+1}(z'') \subset \Xi_j(z')$. Here $j \leq 2s'$ because for $j > 2s'$,

$$\Xi_j(z') \subset \Omega(z'),$$

and by (9.2),

$$\tilde{\mu}(\Omega(z')) \leq \beta^{-15} e^{-2s'} e^{-2d_i s'},$$

which implies (since $\Theta'(z)$ is heavy) that (9.4) can not hold for $\Xi_j(z')$ and $\Theta'(z)$.

Definition 9.6. Let s' be a positive number, and let $\Pi(x)$ and $\tilde{\mu}$ be as in Definition 8.7. To fit our case, let us assume that for any $y \in \mathfrak{J}$,

$$\mathcal{W}(y) = \tilde{\mu}(\Omega'(y)) \leq e^{-2d_i s'} e^{-2s'}.$$

By Lemma 8.15, for every $y \in \mathfrak{J}$ and $z \in \mathcal{I}$ from the same cross-section Σ^r , there is a corresponding neighborhood $\mathcal{C}(\mathcal{T}(y), z) \subset \mathfrak{C}$.

Let us consider the summation

$$\mathfrak{S}(\mathcal{T}(y), z) := \sum_{\substack{y' \in \mathcal{I}_z \\ \mathfrak{D}_z(y') \in \mathcal{C}(\mathcal{T}(y), z) \cap \mathfrak{B}(\mathcal{T}(y), z)}} \mathcal{W}(y').$$

Let j be such that the size of the sides of $\|(v_1, w_1)\|$ are in $[e^{s'-j}, e^{s'-j+1}]$ (cf. Lemma 8.15, particularly equation (8.21)). Then it is easy to see that

$$(9.6) \quad \mathfrak{S}(\mathcal{T}(y), z) \leq N_2 \beta^{-100} \beta^{-100} e^{2s'-j} e^{-2d_i s'} e^{-2s'} \leq \beta^{-201} e^{-2d_i s'-j},$$

where N_2 is a large numeric constant (in [Section 5](#), it was chosen to be 10^6). The curve neighborhood $\mathcal{C}(\mathcal{T}(y), z)$ is called *highly concentrated* if

$$\mathfrak{S}(\mathcal{T}(y), z) \geq e^{-2d_i s' - 7C_1 \varepsilon_1 s'}.$$

Remark 9.7. Note that $\mathfrak{S}(\mathcal{T}(y), z)$ is essentially the $\tilde{\mu}$ -measure of a subset of $\Theta'(z)$ with r such that $z, y \in \Sigma^r$ (in particular, up to the edge effects, $\Theta'(z)$ is a subset of Σ^r). Thus,

$$\mathfrak{S}(\mathcal{T}(y), z) \lesssim \tilde{\mu}(\Sigma^r) \leq e^{-2s'}.$$

Thus, if $d_i < 1$, there can be no highly concentrated curve neighborhoods. Moreover, if $\Theta'(z)$ is not extremely heavy, then, unless $d_i > 2 - O(\varepsilon_1)$, the curve neighborhood $\mathcal{C}(\mathcal{T}(y), z)$ is not highly concentrated.

We further decompose $\bar{\mathcal{R}}_i$ as follows:

Definition 9.8. Given $z \in \mathcal{I}$ such that $\Theta'(z)$ is heavy and normal, and given Σ^r such that $z \in \Sigma^r$, let us define:

$$(9.7) \quad \mathcal{HS}(z, r) := \{y \in \mathfrak{J} \cap \Sigma^r : \mathcal{C}(\mathcal{T}(y), z) \subset \mathfrak{C} \text{ is highly concentrated}\}$$

and

$$(9.8) \quad \mathcal{SS}(z, r) := \bigcup_{y \in \mathcal{HS}(z, r)} \Omega'(y).$$

If there exists Σ^r containing z such that

$$\tilde{\mu}(\mathcal{SS}(z, r)) \geq e^{-2s' - 10C_1 \varepsilon_1 s'},$$

we will call $\Theta'(z)$ *highly structured*. Then we have the following decomposition for $\bar{\mathcal{R}}_i$:

$$(9.9) \quad \bar{\mathcal{R}}_i = \bar{\mathcal{R}}_i^{\mathfrak{r}} \cup \bar{\mathcal{R}}_i^{\mathfrak{s}},$$

where $\bar{\mathcal{R}}_i^{\mathfrak{s}}$ denotes the union of heavy, normal, and highly structured $\Theta'(z)$, and $\bar{\mathcal{R}}_i^{\mathfrak{r}}$ denotes the union of heavy, normal $\Theta'(z)$ that are not highly structured.

We first prove that $\tilde{\mu}(\bar{\mathcal{R}}_i^{\mathfrak{r}})$ is small. The proof that $\tilde{\mu}(\bar{\mathcal{R}}_i^{\mathfrak{s}})$ is small is deferred to the next section.

PROPOSITION 9.9.

$$\tilde{\mu}(\bar{\mathcal{R}}_i^{\mathfrak{r}}) \leq \beta^{22}.$$

Proof. For a contradiction, let us assume $\tilde{\mu}(\bar{\mathcal{R}}_i^{\mathfrak{r}}) \geq \beta^{22}$.

For every $y \in \bar{\mathcal{R}}_i^{\mathfrak{r}}$, if

$$|u(\mathcal{I}(y))y \cap \bar{\mathcal{R}}_i^{\mathfrak{r}}| \geq \beta^{31} e^{2s'},$$

we will call it a *good point*. Otherwise, we call it a *bad point*. Then it is easy to see that the collection of bad points has $\tilde{\mu}$ -measure $\leq \beta^{30}$. Therefore, the collection of good points in $\bar{\mathcal{R}}_i^r$, denoted by $\mathcal{R}'_i$, has $\tilde{\mu}$ -measure $\geq \beta^{22.1}$.

For $r \in [\beta e^{2s'}]$, we say that the cross-section $\Sigma^r \subset \Pi(x)$ is a *good cross-section* if

$$\tilde{\mu}_\tau(\mathcal{R}'_i \cap \Sigma^r) \geq \beta^{30} \tilde{\mu}_\tau(\Sigma^r).$$

Otherwise, we call it a *bad cross-section*. Note that the $\tilde{\mu}_\tau$ -measure of $\mathcal{R}'_i$ contained in bad cross-sections does not exceed β^{30} . Thus, under our assumption that $\tilde{\mu}(\mathcal{R}'_i) \geq \beta^{22.1}$, there must be at least one good cross-section.

Now let us fix a good cross-section Σ^r . We define $\mathcal{S}(\Sigma^r)$ as the number of pairs of tubes from such cross-section that intersect somewhere in the playground:

$$\mathcal{S}(\Sigma^r) := \sum_{(y_1, y_2) \in \diamond_r} \mathcal{W}(y_1) \mathcal{W}(y_2)$$

where $(y_1, y_2) \in \diamond_r$ if the following holds:

- $y_1, y_2 \in \mathfrak{J} \cap \Sigma^r$, and
- $\exists r' \in I(y_1, y_2)$ with $|r'| \geq \beta^{100} e^{2s'}$, and
- $(\Omega'(y_1) \cap \bar{\mathcal{R}}_i^r) \cup (\Omega'(y_2) \cap \bar{\mathcal{R}}_i^r) \neq \emptyset$.

The summands $\mathcal{W}(y_1) \mathcal{W}(y_2)$ can be interpreted as the weight given to the intersection $\mathcal{T}(y_1) \cap \mathcal{T}(y_2)$.

For every $y \in \mathfrak{J} \cap \Sigma^r$ such that $\Omega'(y) \cap \mathcal{R}'_i \neq \emptyset$, we can choose $\hat{y} \in \Omega'(y) \cap \mathcal{R}'_i$. By definition of $\mathcal{R}'_i$, we have that

$$|u(\mathcal{I}(\hat{y}))\hat{y} \cap \bar{\mathcal{R}}_i^r| \geq \beta^{31} e^{2s'}.$$

For every $y' = u(r_{y'})\hat{y} \in u(\mathcal{I}(\hat{y}))\hat{y} \cap \bar{\mathcal{R}}_i^r$ such that $|r_{y'}| \geq \beta^{100} e^{2s'}$, there exists $z \in \mathcal{I}$ such that $y' \in \Theta'(z)$ and

$$\tilde{\mu}(\Theta'(z)) \geq e^{-2s'} e^{-(d_i + \varepsilon_1)s'}.$$

For every $\tilde{z} \in \mathcal{I}_z$, let $r_{\tilde{z}}$ be such that $u(r_{\tilde{z}})\tilde{z} \in \Sigma^r$. Let us define

$$\mathfrak{J}(\tilde{z}, r) := \{\hat{z} \in \mathfrak{J} \cap \Sigma^r : \Omega'(\hat{z}) \cap \Omega(\tilde{z}, r_{\tilde{z}}) \neq \emptyset\},$$

where $\Omega(\tilde{z}, r_{\tilde{z}})$ is as in (8.18). Then for every $\hat{z} \in \mathfrak{J}(\tilde{z}, r)$, we have that $y, \hat{z}$ satisfies $\diamond_r$. Since

$$\Omega(\tilde{z}, r_{\tilde{z}}) \subset \bigcup_{\hat{z} \in \mathfrak{J}(\tilde{z}, r)} \Omega'(\hat{z})$$

and

$$\tilde{\mu}_\tau(\Omega'(\tilde{z})) \asymp \tilde{\mu}_\tau(\Omega(\tilde{z}, r_{\tilde{z}})),$$

we have that

$$\sum_{\hat{z} \in \mathfrak{J}(\tilde{z}, r)} \mathcal{W}(y) \mathcal{W}(\hat{z}) \gg \mathcal{W}(y) \mathcal{W}(\tilde{z}).$$

Let us only consider $\tilde{z} \in \mathcal{I}_{z;y'} := \mathcal{I}_z \cap (\Theta'(z) \setminus \Xi_{C_1\varepsilon_1 s'}(y'))$. Then $\tilde{z} \in \mathcal{I}_{z;y'}$ implies that for any $\hat{z} \in \mathfrak{J}(\tilde{z}, r)$,

$$(9.10) \quad \|\Delta(y, \hat{z})\| \gtrsim \beta e^{-(1+C_1\varepsilon_1)s'}.$$

Note that $\tilde{\mu}_\tau(\Theta'(z) \setminus \Xi_{C_1\varepsilon_1 s'}(y')) \geq \beta \tilde{\mu}_\tau(\Theta'(z))$ by the definition of $\bar{\mathcal{R}}_i^\tau$. By running over all $\tilde{z} \in \mathcal{I}_{z;y'}$, we have that

$$\begin{aligned} \sum_{\tilde{z} \in \mathcal{I}_{z;y'}} \sum_{\hat{z} \in \mathfrak{J}(\tilde{z}, r)} \mathcal{W}(y) \mathcal{W}(\hat{z}) &\gg \sum_{\tilde{z} \in \mathcal{I}_{z;y'}} \mathcal{W}(y) \mathcal{W}(\tilde{z}) \\ &\geq \beta \mathcal{W}(y) \tilde{\mu}(\Theta'(z)) \geq \beta \mathcal{W}(y) e^{-2s'} e^{-(d_i + \varepsilon_1)s'}. \end{aligned}$$

Since

$$|u(\mathfrak{l}(\hat{y}))\hat{y} \cap \bar{\mathcal{R}}_i^\tau| \geq \beta^{31} e^{2s'},$$

we can find a set $\mathcal{Y}$ of at least $\beta^{30} e^{2s'}$ different $y' \in u(\mathfrak{l}(\hat{y}))\hat{y} \cap \bar{\mathcal{R}}_i$ that are β -separated. The corresponding summation is:

$$\begin{aligned} \sum_{y' \in \mathcal{Y}} \sum_{\tilde{z} \in \mathcal{I}_{z;y'}} \sum_{\hat{z} \in \mathfrak{J}(\tilde{z}, r)} \mathcal{W}(y) \mathcal{W}(\hat{z}) &\gg \sum_{y' \in \mathcal{Y}} \beta \mathcal{W}(y) e^{-2s'} e^{-(d_i + \varepsilon_1)s'} \\ &\geq \beta^{31} e^{2s'} \mathcal{W}(y) e^{-2s'} e^{-(d_i + \varepsilon_1)s'} \\ &= \beta^{31} \mathcal{W}(y) e^{-(d_i + \varepsilon_1)s'}. \end{aligned}$$

By (9.10) and Lemma 8.14, for every $\hat{z} \in \mathfrak{J} \cap \Sigma^r$ such that $(y, \hat{z}) \in \diamond_r$ and (9.10), we have that $I(\hat{y}, \hat{z}) \subset I$, where I is an interval of length that does not exceed $e^{C_1\varepsilon_1 s'}$. Note that the summand $\mathcal{W}(y) \mathcal{W}(\hat{z})$ appears in the summation for $y' \in \mathcal{Y}$ only if $y' = u(r_{y'})\hat{y}$ where $r_{y'} \in I(\hat{y}, \hat{z})$. Therefore, every summand $\mathcal{W}(y) \mathcal{W}(\hat{z})$ from above is counted in summations for at most $\beta^{-1} e^{C_1\varepsilon_1 s'}$ different $y' \in \mathcal{Y}$. Therefore, we have that the contribution to $\mathcal{S}(\Sigma^r)$ from the above summation is at least $\beta^{32} \mathcal{W}(y) e^{-(d_i + \varepsilon_1)s' - C_1\varepsilon_1 s'}$. By running over all $y \in \mathfrak{J} \cap \Sigma^r$ such that $\Omega'(y) \cap \bar{\mathcal{R}}'_i \neq \emptyset$, we get

$$(9.11) \quad \mathcal{S}(\Sigma^r) \geq \beta^{62} \tilde{\mathfrak{m}} e^{-(d_i + \varepsilon_1)s' - C_1\varepsilon_1 s'} \geq \tilde{\mathfrak{m}} e^{-d_i s' - 2C_1\varepsilon_1 s'},$$

where $\tilde{\mathfrak{m}} = \tilde{\mu}_\tau(\Sigma^r) \asymp e^{-2s'}$.

On the other hand, let us get an upper bound on $\mathcal{S}(\Sigma^r)$.

For any fixed $y \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I} \cap \Sigma^r \cap \bar{\mathcal{R}}_i^\tau$, let us consider all $z' \in \mathcal{I}_z$ such that there exists $r' \in I(y, z')$ with $|r'| \geq \beta^{100} e^{2s'}$. By Lemma 8.15, $\mathfrak{D}_z(z') \in \mathcal{C}(\mathcal{T}(y), z)$.

Let us first consider the case $y \in \mathcal{HS}(z, r)$. By (9.6), we have that

$$\begin{aligned} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') &\leq \beta^{-101} \mathcal{W}(y) e^{2s'} e^{-2d_i s'} e^{-2s'} \\ &= \mathcal{W}(y) e^{-2d_i s' + \varepsilon_1 s'}. \end{aligned}$$

From the definition of $\bar{\mathcal{R}}_i^{\mathfrak{r}}$, we have that

$$\sum_{y \in \mathcal{HS}(z, r)} \mathcal{W}(y) \ll \tilde{\mu}(\mathcal{SS}(z, r)) \leq e^{-2s' - 10C_1\varepsilon_1 s'}.$$

Thus, by summing over $y \in \mathcal{HS}(z, r)$, we have

$$\begin{aligned} \sum_{y \in \mathcal{HS}(z, r)} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') &\leq \sum_{y \in \mathcal{HS}(z, r)} \mathcal{W}(y) e^{-2d_i s' + \varepsilon_1 s'} \\ &\leq e^{-2s' - 10C_1\varepsilon_1 s'} e^{-2d_i s' + \varepsilon_1 s'} \leq e^{-2s' - 2d_i s' - 9C_1\varepsilon_1 s'}. \end{aligned}$$

We then consider the case $y \notin \mathcal{HS}(z, r)$. Since in this case $\mathcal{C}(\mathcal{T}(y), z)$ is not highly concentrated, we have that

$$\begin{aligned} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') &\leq \mathcal{W}(y) e^{2s' - 7C_1\varepsilon_1 s'} e^{-2d_i s'} e^{-2s'} \\ &= \mathcal{W}(y) e^{-2d_i s' - 7C_1\varepsilon_1 s'}. \end{aligned}$$

Note that

$$\sum_{y \notin \mathcal{HS}(z, r)} \mathcal{W}(y) \ll \tilde{\mu}(\Sigma^r) \ll e^{-2s'}.$$

By summing over $y \notin \mathcal{HS}(z, r)$, we have

$$\begin{aligned} \sum_{y \notin \mathcal{HS}(z, r)} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') &\leq \sum_{y \notin \mathcal{HS}(z, r)} \mathcal{W}(y) e^{-2d_i s' - 7C_1\varepsilon_1 s'} \\ &\leq e^{-2s'} e^{-2d_i s' - 7C_1\varepsilon_1 s'} = e^{-2s' - 2d_i s' - 7C_1\varepsilon_1 s'}. \end{aligned}$$

Therefore, by summing over $y \in \mathfrak{J} \cap \Sigma^r$, we have

$$\begin{aligned} \sum_{y \in \mathfrak{J} \cap \Sigma^r} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') &\leq e^{-2s' - 2d_i s' - 9C_1\varepsilon_1 s'} + e^{-2s' - 2d_i s' - 7C_1\varepsilon_1 s'} \\ &\leq e^{-2s' - 2d_i s' - 6.9C_1\varepsilon_1 s'}. \end{aligned}$$

Since $z \in \mathcal{R}_i$, we have

$$\tilde{\mu}_{\mathfrak{r}}(\Theta'(z)) \geq e^{-(d_i + \varepsilon_1)s'} e^{-2s'}.$$

Thus, we have that there are at most $\tilde{\mathfrak{m}} e^{2s'} e^{d_i s' + \varepsilon_1 s'}$ different z from $\mathcal{I} \cap \Sigma^r \cap \mathcal{R}_i$.

By running over all $z \in \mathcal{I} \cap \mathcal{R}_i \cap \Sigma^r$, we have

$$\begin{aligned} \mathcal{S}(\Sigma^r) &= \sum_{z \in \mathcal{I} \cap \mathcal{R}_i \cap \Sigma^r} \sum_{y \in \mathfrak{J} \cap \Sigma^r} \sum_{z' \in \mathcal{I}_z, (y, z') \in \diamond_r} \mathcal{W}(y) \mathcal{W}(z') \\ &\leq \sum_{z \in \mathcal{I} \cap \mathcal{R}_i \cap \Sigma^r} e^{-2s' - 2d_i s' - 6.9C_1\varepsilon_1 s'} \\ &\leq \tilde{\mathfrak{m}} e^{2s'} e^{d_i s' + \varepsilon_1 s'} e^{-2s' - 2d_i s' - 6.9C_1\varepsilon_1 s'} = \tilde{\mathfrak{m}} e^{-d_i s' - 6C_1\varepsilon_1 s'}. \end{aligned}$$

This upper bound is smaller than the lower bound in (9.11) which leads to a contradiction. This proves that $\tilde{\mu}(\bar{\mathcal{R}}_i^{\mathfrak{r}}) < \beta^{22}$. $\square$

Using arguments similar to the proof of [Proposition 9.9](#), we will prove that the measure of $\mathcal{R}_{i,j}$ is small:

PROPOSITION 9.10. *For any $C_1\varepsilon_1 s' \leq j \leq 2s'$, we have*

$$\tilde{\mu}(\mathcal{R}_{i,j}) \leq \beta^{22}.$$

Proof. For each $\Theta'(\bar{z}) \subset \mathcal{R}_{i,j}$, fix a subset $\Xi_j(z) \subset \Theta'(\bar{z})$ as in (9.4), and let $\mathcal{R}'_{i,j}$ denote the union of all of those $\Xi_j(z)$ for $\Theta'(\bar{z}) \subset \mathcal{R}_{i,j}$. To prove $\tilde{\mu}_\tau(\mathcal{R}_{i,j}) < \beta^{22}$, it suffices to show that $\tilde{\mu}_\tau(\mathcal{R}'_{i,j}) < \beta^{22.1}$.

Assume in contradiction that $\tilde{\mu}_\tau(\mathcal{R}'_{i,j}) \geq \beta^{22.1}$. Using similar arguments as in the proof of [Proposition 9.9](#), we can get a subset $\mathcal{R}''_{i,j} \subset \mathcal{R}'_{i,j}$ such that $\tilde{\mu}_\tau(\mathcal{R}''_{i,j}) \geq \beta^{22.2}$ and for any $y \in \mathcal{R}''_{i,j}$,

$$|u(\mathbf{l}(y))y \cap \mathcal{R}'_{i,j}| \geq \beta^{30}e^{2s'},$$

and get a cross-section Σ^r such that

$$\tilde{\mu}_\tau(\Sigma^r \cap \mathcal{R}''_{i,j}) \geq \beta^{30}\tilde{\mu}_\tau(\Sigma^r).$$

Let us define $\mathcal{S}_j(\Sigma^r)$ as follows:

$$\mathcal{S}_j(\Sigma^r) := \sum_{(y_1, y_2) \in \diamond_{r,j}} \mathcal{W}(y_1)\mathcal{W}(y_2),$$

where $(y_1, y_2) \in \diamond_{r,j}$ if

- $y_1, y_2 \in \mathfrak{J} \cap \Sigma^r$, and
- $\exists r' \in I(y_1, y_2)$ with $|r'| \geq \beta^{100}e^{2s'}$, and
- $(\Omega'(y_1) \cap \mathcal{R}'_{i,j}) \cup (\Omega'(y_2) \cap \mathcal{R}'_{i,j}) \neq \emptyset$, and
- $\|\Delta(y_1, y_2)\| \in [\beta e^{-s'-j-1}, \beta e^{-s'-j}]$.

Let us first get a lower bound on $\mathcal{S}_j(\Sigma^r)$. For any fixed $y \in \mathfrak{J} \cap \Sigma^r$ with $\Omega'(y) \cap \mathcal{R}''_{i,j} \neq \emptyset$, we can find $\hat{y} \in \Omega'(y) \cap \mathcal{R}''_{i,j}$. By the properties of $\mathcal{R}''_{i,j}$, we have

$$|u(\mathbf{l}(\hat{y}))\hat{y} \cap \mathcal{R}'_{i,j}| \geq \beta^{22}e^{2s'}.$$

For any $y' \in u(\mathbf{l}(\hat{y}))\hat{y} \cap \mathcal{R}'_{i,j}$, there exists $z \in \mathcal{I}$ and $\Xi_j(z') \subset \Theta'(z)$ with $y' \in \Xi_j(z')$ such that

$$\tilde{\mu}_\tau(\Xi_j(z')) \geq (1 - \beta)^j e^{-2s'} e^{-(d_i + \varepsilon_1)s'}$$

and

$$\tilde{\mu}_\tau(\Xi_{j+1}(y')) < (1 - \beta)\tilde{\mu}_\tau(\Xi_j(z')).$$

Let us denote $\mathcal{I}_{y',j} := \mathcal{I}_z \cap (\Xi_j(z') \setminus \Xi_{j+1}(y'))$. Following the same argument as in the proof of [Proposition 9.9](#), for every $y'_2 \in \mathcal{I}_{y',j}$ we can get a finite subset $\mathfrak{J}_j(y'_2, r) \subset \mathfrak{J} \cap \Sigma^r$ such that for any $y_2 \in \mathfrak{J}_j(y'_2, r)$, (y_2, y') satisfies $\diamond_{r,j}$, and

$$\sum_{y_2 \in \mathfrak{J}_j(y'_2, r)} \mathcal{W}(y)\mathcal{W}(y_2) \gg \mathcal{W}(y)\mathcal{W}(y'_2).$$

By running over all $y'_2 \in \mathcal{I}_{y',j}$, we get

$$\begin{aligned}
& \sum_{y'_2 \in \mathcal{I}_{y',j}} \sum_{y_2 \in \mathfrak{J}_j(y'_2, r)} \mathcal{W}(y) \mathcal{W}(y_2) \\
& \gg \sum_{y'_2 \in \mathcal{I}_{y',j}} \mathcal{W}(y) \mathcal{W}(y'_2) \gg \mathcal{W}(y) \tilde{\mu}_\tau(\Xi_j(z') \setminus \Xi_{j+1}(y')) \\
& > \mathcal{W}(y) \beta (1 - \beta)^j e^{-2s'} e^{-(d_i + \varepsilon_1)s'} > \mathcal{W}(y) e^{-2s'} e^{-(d_i + 2\varepsilon_1)s'}.
\end{aligned}$$

Similarly to the proof of the lower bound for $\mathcal{S}(\Sigma^r)$ in the proof of [Proposition 9.9](#) (specifically, the discussion in [p. 229](#)), since

$$|u(\mathbf{l}(\hat{y}))\hat{y} \cap \mathcal{R}'_{i,j}| \geq \beta^{30} e^{2s'},$$

we can find a set $\mathcal{Y}_j$ of at least $\beta^{29} e^{2s'}$ different $y' \in u(\mathbf{l}(\hat{y}))\hat{y} \cap \mathcal{R}'_{i,j}$ that are β -separated. The corresponding summation is:

$$\begin{aligned}
(9.12) \quad & \sum_{y' \in \mathcal{Y}_j} \sum_{y'_2 \in \mathcal{I}_{y',j}} \sum_{y_2 \in \mathfrak{J}_j(y'_2, r)} \mathcal{W}(y) \mathcal{W}(y_2) \\
& \gg \sum_{y' \in \mathcal{Y}_j} \mathcal{W}(y) e^{-2s'} e^{-(d_i + 2\varepsilon_1)s'} \\
& \gg \beta^{29} e^{2s'} \mathcal{W}(y) e^{-2s'} e^{-(d_i + 2\varepsilon_1)s'} = \beta^{29} \mathcal{W}(y) e^{-(d_i + 2\varepsilon_1)s'}.
\end{aligned}$$

Since $\|\Delta(y, y_2)\| \geq \beta e^{-s' - j - 1}$, by [Lemma 8.14](#), for each $y_2 \in \mathfrak{J} \cap \Sigma^r$ such that $(y, y_2) \in \diamond_{r,j}$, we have that $I(y, y_2) \subset I$, where I is an interval of length less than or equal to e^{j+1} . Note that the summand $\mathcal{W}(y) \mathcal{W}(y_2)$ appears in the summation (9.12) for $y' \in \mathcal{Y}_j$ only if $y' = u(r_{y'})\hat{y}$, where $r_{y'} \in I(y, y_2)$. Therefore, every summand $\mathcal{W}(y) \mathcal{W}(y_2)$ appears in (9.12) for at most $\beta^{-1} e^{j+1}$ different $y' \in \mathcal{Y}_j$. This implies that for a fixed $y \in \mathfrak{J} \cap \Sigma^r \cap \mathcal{R}''_{i,j}$,

$$\begin{aligned}
& \sum_{y_2: (y, y_2) \in \diamond_{r,j}} \mathcal{W}(y) \mathcal{W}(y_2) \geq \beta^{30} \mathcal{W}(y) e^{-(d_i + 2\varepsilon_1)s' - j - 1} \\
& > \mathcal{W}(y) e^{-(d_i + 3\varepsilon_1)s' - j}.
\end{aligned}$$

By summing over all $y \in \mathfrak{J} \cap \Sigma^r$ with $\Omega'(y) \cap \mathcal{R}''_{i,j} \neq \emptyset$, we have

$$(9.13) \quad \mathcal{S}_j(\Sigma^r) \geq \tilde{\mu}(\mathcal{R}''_{i,j} \cap \Sigma^r) e^{-(d_i + 3\varepsilon_1)s' - j} \geq \beta^{30} \tilde{\mathbf{m}} e^{-(d_i + 3\varepsilon_1)s' - j},$$

where $\tilde{\mathbf{m}} = \tilde{\mu}(\Sigma^r)$.

Now let us get an upper bound on $\mathcal{S}_j(\Sigma^r)$. To do this, we need the following lemma, for a fixed $y_2 \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I}$ such that $\Theta'(z)$ contains $\Xi_j(z') \subset \Sigma^r \cap \mathcal{R}'_{i,j}$:

LEMMA 9.11. *If there is at least one $y_1 \in \mathcal{I}_z \cap \Xi_j(z')$ such that $(y_2, y_1) \in \diamond_{r,j}$, then*

$$z' \in \Sigma_{s', j-1}(y_2).$$

Proof. Since $z', y_2 \in \Sigma^r$, it suffices to show that

$$z' \in \Pi_{s', j-1}(y_2).$$

We write

$$y_1 = a_0 a u^*(r_{y_1}^*) \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) u(r_{y_1}) y_2,$$

where $\mathbf{w}^- \in \mathbf{r}^-$ satisfies $\|\mathbf{w}^-\| \in [\beta e^{-s'-j-1}, \beta e^{-s'-j}]$. Then, by Lemma 8.15, we have $\|\mathbf{w}^+\| \leq \beta e^{s'-j}$. Let $\hat{y}_1 = a(-2t)y_1$, $\hat{y}_2 = a(-2t)y_2$. Then we have

$$\hat{y}_1 = a_0 a u^*(\hat{r}^*) \exp(\hat{\mathbf{w}}^-) \exp(\hat{\mathbf{w}}^+) u(\hat{r}) \hat{y}_2,$$

where $\hat{r}^*, \hat{r} \in [\beta]$, $\hat{\mathbf{w}}^- \in B_{\mathbf{r}^-}(\beta e^{-j})$, and $\hat{\mathbf{w}}^+ \in B_{\mathbf{r}^+}(\beta e^{-j})$. By Lemma 2.3,

$$\exp(\hat{\mathbf{w}}^-) \exp(\hat{\mathbf{w}}^+) = \exp(\hat{\mathbf{w}}) \exp(\hat{\mathbf{h}}),$$

where $\hat{\mathbf{w}} \in B_{\mathbf{r}_1 + \mathbf{r}_2}(1.1\beta e^{-j})$, and $\hat{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathbf{h}}(0.1\beta e^{-2j})$. Therefore,

$$(9.14) \quad \hat{y}_1 = \exp(\tilde{\mathbf{w}}) \exp(\tilde{\mathbf{h}}) \hat{y}_2,$$

where $\tilde{\mathbf{w}} \in B_{\mathbf{r}_1 + \mathbf{r}_2}(1.3\beta e^{-j})$ and $\tilde{\mathbf{h}} \in B_{\mathbb{R}\mathbf{a}_0 + \mathbf{h}}(1.1\beta)$. Note that $y_1 \in \Xi_j(z')$; we have

$$y_1 = \exp(\mathbf{w}_1^+ + \mathbf{w}_1^-) \exp(\mathbf{h}_1) z',$$

where $\mathbf{w}_1^+ \in B_{\mathbf{r}^+}(\beta e^{-s'})$, $\mathbf{w}_1^- \in B_{\mathbf{r}^-}(\beta e^{-s'-j})$, and $\mathbf{h}_1 \in \mathfrak{q}^{A_0 H}(2s', \beta)$. Let us denote $\hat{z}' = a(-2s')z'$; then we have

$$(9.15) \quad \hat{y}_1 = \exp(\tilde{\mathbf{w}}_1) \exp(\tilde{\mathbf{h}}_1) \hat{z}',$$

where $\tilde{\mathbf{w}}_1 \in B_{\mathbf{r}_1 + \mathbf{r}_2}(\beta e^{-j})$ and $\tilde{\mathbf{h}}_1 \in B_{\mathbb{R}\mathbf{a}_0 + \mathbf{h}}(\beta)$. Equations (9.14) and (9.15) imply that

$$\hat{y}_2 = \exp(-\tilde{\mathbf{h}}) \exp(-\tilde{\mathbf{w}}) \exp(\tilde{\mathbf{w}}_1) \exp(\tilde{\mathbf{h}}_1) \hat{z}'.$$

By Lemma 2.3, we have

$$\hat{y}_2 = \exp(\tilde{\mathbf{w}}_2) \exp(\tilde{\mathbf{h}}_2) \hat{z}',$$

where $\tilde{\mathbf{w}}_2 \in B_{\mathbf{r}_1 + \mathbf{r}_2}(2.1\beta e^{-j})$ and $\tilde{\mathbf{h}}_2 \in B_{\mathbb{R}\mathbf{a}_0 + \mathbf{h}}(2.1\beta)$. Noting that $\hat{y}_2 = a(-2s')y_2$, and $\hat{z}' = a(-2s')z'$, we have

$$z' \in \Pi_{s', j-1}(y_2).$$

This completes the proof of the lemma. $\square$

We now continue with the proof of Proposition 9.10. Recall that $y_2 \in \mathfrak{J} \cap \Sigma^r$, and $z \in \mathcal{I}$ is such that $\Theta'(z)$ contains a $\Xi_j(z') \subset \Sigma^r \cap \mathcal{R}'_{i,j}$. Using similar arguments to those used in the proof of Lemma 9.11, we can prove that

$$\Xi_j(z') \subset \Sigma_{s', j-2}(y_2).$$

Define

$$(9.16) \quad \mathcal{I}(y_2, j) := \left\{ z \in \mathcal{I} \cap \Sigma^r \cap \mathcal{R}_{i,j} : \begin{array}{l} \text{the heavy slice } \Xi_j(z') \subset \Theta'(z) \\ \text{satisfies } \Xi_j(z') \subset \Sigma_{s', j-2}(y_2) \cap \mathcal{R}'_{i,j} \end{array} \right\}.$$

Fix $z \in \mathcal{I}(y_2, j)$. By Lemma 8.15, the coordinates $\mathfrak{D}_z(y_1)$ of all $y_1 \in \mathcal{I}_z$ such that $(y_2, y_1) \in \diamond_{r,j}$ are contained in

$$\mathcal{C}(\mathcal{T}(y_2), z) \cap \mathfrak{B}(\mathcal{T}(y_2), z).$$

By $\diamond_{r,j}$ and Lemma 8.15, we have that if there is at least one such y_1 the size of $\mathfrak{B}(\mathcal{T}(y_2), z)$ does not exceed $\beta^{-98}e^{-s'-j}$.

Using (9.6), we may conclude that

$$\sum_{\mathfrak{D}_z(y_1) \in \mathcal{C}(\mathcal{T}(y_2), z) \cap \mathfrak{B}(\mathcal{T}(y_2), z)} \mathcal{W}(y_1)\mathcal{W}(y_2) \ll \mathcal{W}(y_2)\beta^{-198}e^{-j}e^{-2d_is'}.$$

By Lemma 8.6, we have

$$\mu(\Sigma_{s',j-2}(y_2)) \leq e^{-2s'}e^{-d_0(j-2)}.$$

Since $\Pi(x)$ is a full playground, we have

$$\tilde{\mu}(\Sigma_{s',j-2}(y_2)) \leq \beta^{-15}e^{-2s'}e^{-d_0(j-2)}.$$

Noting that for z' as in (9.16)

$$\tilde{\mu}_\tau(\Xi_j(z')) \geq (1 - \beta)^j e^{-2s'} e^{-(d_i + \varepsilon_1)s'},$$

we have that

$$\begin{aligned} |\mathcal{I}(y_2, j)| &\leq \frac{\beta^{-15}e^{-2s'}e^{-d_0(j-2)}}{(1 - \beta)^j e^{-2s'} e^{-(d_i + \varepsilon_1)s'}} \\ &= \beta^{-15}(1 - \beta)^{-j} e^{-d_0(j-2)} e^{(d_i + \varepsilon_1)s'}. \end{aligned}$$

Thus, for a fixed $y_2 \in \mathfrak{J} \cap \Sigma^r$, the corresponding sum is as follows:

$$\begin{aligned} &\sum_{z \in \mathcal{I}(y_2, j)} \sum_{\substack{\mathfrak{D}_z(y_1) \in \mathcal{C}(\mathcal{T}(y_2), z) \\ \cap \mathfrak{B}(\mathcal{T}(y_2), z)}} \mathcal{W}(y_1)\mathcal{W}(y_2) \\ &\ll \mathcal{W}(y_2)\beta^{-198}e^{-j}e^{-2d_is'}\beta^{-15}(1 - \beta)^{-j}e^{-d_0(j-2)}e^{(d_i + \varepsilon_1)s'} \\ &\leq 2\mathcal{W}(y_2)\beta^{-213}e^{-d_0(j-2)}e^{(d_i + \varepsilon_1)s'}e^{-j}e^{-2d_is'}. \end{aligned}$$

By summing over all $y_2 \in \mathfrak{J} \cap \Sigma^r$, we get

$$\mathcal{S}_j(\Sigma^r) \leq 2\tilde{\mathfrak{m}}\beta^{-213}e^{-d_0(j-2)}e^{(d_i + \varepsilon_1)s'}e^{-j}e^{-2d_is'} \leq \tilde{\mathfrak{m}}e^{2\varepsilon_1s'}e^{-d_0j}e^{-d_is'-j}.$$

This upper bound is smaller than the lower bound in (9.13) because $d_0j \geq d_0C_1\varepsilon_1s' \geq 10\varepsilon_1s'$. This leads to a contradiction, completing the proof of Proposition 9.10. $\square$

10. No loss of dimension in a coarser scale

Using similar arguments to those in the proofs of [Proposition 9.9](#) and [Proposition 9.10](#), we can also prove that, compared to μ_i , the dimension of $\tilde{\mu}$ does not drop significantly:

PROPOSITION 10.1. *Let us keep the notation as in [Proposition 9.1](#). Then there exists $\mathfrak{Z} \subset \Pi(x)$ with $\tilde{\mu}(\mathfrak{Z}) \leq \beta^{50}$ such that for $x \in \mathcal{I} \setminus \mathfrak{Z}$,*

$$(10.1) \quad \tilde{\mu}(\Theta'(x)) \leq e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'}.$$

Sketch of the proof. For $z \in \mathcal{I}$, let us call $\Theta'(z)$ *extremely heavy* if

$$\tilde{\mu}(\Theta'(z)) > \tilde{M}_{s'} \quad \text{for } \tilde{M}_{s'} = e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'}.$$

Let $\mathfrak{Z}$ denote the union of extremely heavy $\Theta'(z)$'s. Then it suffices to show that

$$\tilde{\mu}(\mathfrak{Z}) \leq \beta^{50}.$$

Similarly to [\(9.3\)](#), let us decompose

$$\mathfrak{Z} = \mathfrak{Z}' \cup \bigcup_{C_1 \varepsilon_1 s' \leq j \leq 2s'} \mathfrak{Z}_j.$$

Here $\mathfrak{Z}'$ denotes the union of extremely heavy $\Theta'(z)$ such that for any

$$\Xi_{C_1 \varepsilon_1 s'}(y) \subset \Theta'(z),$$

we have

$$\tilde{\mu}(\Xi_{C_1 \varepsilon_1 s'}(y)) \leq (1 - \beta) \tilde{\mu}(\Theta'(z)),$$

and $\mathfrak{Z}_j$ denotes the union of extremely heavy $\Theta'(z)$ such that there exists $\Xi_j(y) \subset \Theta'(z)$ such that

$$\tilde{\mu}(\Xi_j(y)) > (1 - \beta)^j \tilde{\mu}(\Theta'(z)),$$

and for any $\Xi_{j+1}(y') \subset \Xi_j(y)$,

$$\tilde{\mu}(\Xi_{j+1}(y')) \leq (1 - \beta) \tilde{\mu}(\Xi_j(y)).$$

Then it suffices to show that $\tilde{\mu}(\mathfrak{Z}') \leq \beta^{50.1}$ and

$$\tilde{\mu}(\mathfrak{Z}_j) \leq \beta^{50.1}, \text{ for any } j \in [C_1 \varepsilon_1 s', 2s'].$$

We first sketch the proof of

$$\tilde{\mu}(\mathfrak{Z}') \leq \beta^{50.1},$$

which follows the proof of [Proposition 9.9](#).

Assume in contradiction that

$$\tilde{\mu}(\mathfrak{Z}') > \beta^{50.1}.$$

By repeating the argument in the proof of [Proposition 9.9](#), we get $\mathfrak{Z}'' \subset \mathfrak{Z}'$ and a cross-section Σ^r such that the following holds:

- $\tilde{\mu}(\mathfrak{Z}'') \geq \beta^{51}$;
- $\tilde{\mu}(\mathfrak{Z}'' \cap \Sigma^r) \geq \beta^{52} \tilde{\mu}(\Sigma^r)$;
- For any $y \in \mathfrak{Z}''$, $|u(l(y))y \cap \mathfrak{Z}'| \geq \beta^{52} e^{2s'}$.

Now let us consider the intersection number $\mathcal{Z}(\Sigma^r)$ as follows:

$$\mathcal{Z}(\Sigma^r) := \sum_{(y_1, y_2) \in \diamond'_r} \mathcal{W}(y_1) \mathcal{W}(y_2),$$

where $(y_1, y_2) \in \diamond'_r$ if the following holds:

- $y_1, y_2 \in \mathfrak{J} \cap \Sigma^r$, and
- $\exists r' \in I(y_1, y_2)$ with $|r'| \geq \beta^{100} e^{2s'}$, and
- $(\Omega'(y_1) \cap \mathfrak{Z}') \cup (\Omega'(y_2) \cap \mathfrak{Z}') \neq \emptyset$.

We can repeat the argument in the proof of [Proposition 9.9](#) to get lower and upper bounds on $\mathcal{Z}(\Sigma^r)$ (similarly to $\mathcal{S}(\Sigma^r)$):

For lower bound, we get that, for any $y_1 \in \mathfrak{J} \cap \Sigma^r$ such that

$$\Omega'(y_1) \cap \mathfrak{Z}'' \neq \emptyset,$$

the summation is as follows:

$$(10.2) \quad \sum_{y_2: (y_1, y_2) \in \diamond'_r} \mathcal{W}(y_1) \mathcal{W}(y_2) \gg \beta^{52} e^{2s' - C_1 \varepsilon_1 s'} \beta \tilde{M}_{s'} \mathcal{W}(y_1)$$

since the trajectory $u(l(y))y \cap \mathfrak{Z}'$ encounters approximately more than $\beta^{51} e^{2s'}$ very heavy Θ' neighbourhoods sufficiently far from Σ^r . For each such $\Theta'(z)$,

$$(10.3) \quad \tilde{M}_{s'} \leq \tilde{\mu}(\Theta'(z)) \approx \sum_{y_2}' \mathcal{W}(y_2),$$

where the sum is over all $y_2 \in \mathfrak{J} \cap \Sigma^r$ so that $(y_1, y_2) \in \diamond'_r$, and there are $|r'| \leq \beta e^{2s'}$ and $y'_2 \in \Omega(y_2)$ such that $u(r')y'_2 \in \Theta'(z)$. However, each such (y_1, y_2) may appear in the sum (10.3) up to $|I(y_1, y_2)|\beta^{-1}$ many times. Throwing away all y_2 for which the corresponding $u(r')$ trajectory meets $\Theta'(z)$ in one $\Xi_{C_1 \varepsilon_1 s'}$ -subneighbourhood, we can ensure that $|I(y_1, y_2)| \lesssim e^{C_1 \varepsilon_1 s'}$. Combining everything, one gets (10.2).

By summing up all $y_1 \in \mathfrak{J} \cap \Sigma^r$ with $\Omega'(y_1) \cap \mathfrak{Z}'' \neq \emptyset$, we have

$$\begin{aligned} \mathcal{Z}(\Sigma^r) &\gg \beta^{52} \tilde{\mathfrak{m}} \beta^{52} e^{2s' - C_1 \varepsilon_1 s'} \beta \tilde{M}_{s'} \\ &= \beta^{110} \tilde{\mathfrak{m}} e^{-d_i s'} \geq \tilde{\mathfrak{m}} e^{-d_i s' - \varepsilon_1 s'}, \end{aligned}$$

where $\tilde{\mathfrak{m}} = \tilde{\mu}(\Sigma^r)$.

For the upper bound, by [Lemma 8.15](#), for any $y_1 \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'$, we have that for any $y_2 \in \mathcal{I}_z$ such that $(y_1, y_2) \in \diamond'_r$, we have

$$\mathfrak{D}_z(y_2) \in \mathcal{C}(\mathcal{T}(y_1), z).$$

By (9.6), we get

$$\sum_{y_2 \in \mathcal{I}_z : (y_1, y_2) \in \mathcal{O}'_r} \mathcal{W}(y_1) \mathcal{W}(y_2) \leq \mathcal{W}(y_1) \beta^{-101} e^{-2d_i s'}.$$

Note that for any $z \in \mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'$,

$$\tilde{\mu}(\Theta'(z)) > e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'}.$$

Thus, we have

$$|\mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'| < \tilde{\mathfrak{m}} e^{2s'} e^{(d_i - C_1 \varepsilon_1)s'} \leq e^{(d_i - C_1 \varepsilon_1)s'}.$$

Therefore, by summing over all $y_1 \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'$, we have

$$\begin{aligned} \mathcal{Z}(\Sigma^r) &\leq \tilde{\mathfrak{m}} |\mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'| \beta^{-101} e^{-2d_i s'} \\ &\leq \tilde{\mathfrak{m}} e^{(d_i - C_1 \varepsilon_1)s'} \beta^{-101} e^{-2d_i s'} \leq \tilde{\mathfrak{m}} e^{(-d_i - C_1 \varepsilon_1 + \varepsilon_1)s'}. \end{aligned}$$

The upper bound is smaller than the lower bound, leading to a contradiction. This proves that

$$\tilde{\mu}(\mathfrak{Z}') \leq \beta^{50.1}.$$

We then sketch the proof of

$$\tilde{\mu}(\mathfrak{Z}_j) \leq \beta^{50.1}, \text{ for any } j \in [C_1 \varepsilon_1 s', 2s'],$$

which follows the proof of [Proposition 9.10](#).

For every $z \in \mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}_j$, let us fix a $\Xi_j(z') \subset \Theta'(z)$ such that

$$\tilde{\mu}(\Xi_j(z')) > (1 - \beta)^j e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'},$$

and for any $\Xi_{j+1}(z'') \subset \Xi_j(z')$,

$$\tilde{\mu}(\Xi_{j+1}(z'')) \leq (1 - \beta) \tilde{\mu}(\Xi_j(z'))$$

Let $\mathfrak{Z}'_j$ denote the union of these $\Xi_j(z')$'s. Then it suffices to show that

$$\tilde{\mu}(\mathfrak{Z}'_j) \leq \beta^{50.1}.$$

Let us assume in contradiction that

$$\tilde{\mu}(\mathfrak{Z}'_j) > \beta^{50.1}.$$

Repeating the argument as in the proof of [Proposition 9.9](#), we will get $\mathfrak{Z}''_j \subset \mathfrak{Z}'_j$ and a cross-section Σ^r satisfying the following:

- $\tilde{\mu}(\mathfrak{Z}''_j) \geq \beta^{51}$;
- $\tilde{\mu}(\mathfrak{Z}''_j \cap \Sigma^r) \geq \beta^{52} \tilde{\mu}(\Sigma^r)$;
- For any $y \in \mathfrak{Z}''_j$, we have $|u(\mathbf{l}(y))y \cap \mathfrak{Z}'_j| \geq \beta^{52} e^{2s'}$.

Now let us consider the intersection number $\mathcal{Z}_j(\Sigma^r)$ as follows:

$$\mathcal{Z}_j(\Sigma^r) := \sum_{(y_1, y_2) \in \mathcal{O}'_{r,j}} \mathcal{W}(y_1) \mathcal{W}(y_2),$$

where $(y_1, y_2) \in \mathcal{O}'_{r,j}$ if the following holds:

- $y_1, y_2 \in \mathfrak{J} \cap \Sigma^r$, and
- $\exists r' \in I(y_1, y_2)$ with $|r'| \geq \beta^{100} e^{2s'}$, and
- $\|\Delta(y_1, y_2)\| \in [\beta e^{-s'-j-1}, \beta e^{-s'-j}]$, and
- $(\Omega'(y_1) \cap \mathfrak{Z}') \cup (\Omega'(y_2) \cap \mathfrak{Z}') \neq \emptyset$.

We can repeat the argument in the proof of [Proposition 9.10](#) to get lower and upper bounds on $\mathcal{Z}_j(\Sigma^r)$ (similarly to $\mathcal{S}_j(\Sigma^r)$):

For lower bound, we will get that for any $y_1 \in \mathfrak{J} \cap \Sigma^r$ with

$$\Omega'(y_1) \cap \mathfrak{Z}_j'' \neq \emptyset,$$

the summation satisfies the following:

$$\begin{aligned} \sum_{y_2: (y_1, y_2) \in \diamond'_{r,j}} \mathcal{W}(y_1) \mathcal{W}(y_2) &\gg \mathcal{W}(y_1) \beta^{51} e^{2s'-j} \beta (1-\beta)^j \tilde{M}_{s'} \\ &\geq \mathcal{W}(y_1) \beta^{53} e^{2s'-j} e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'} \\ &\geq \mathcal{W}(y_1) \beta^{53} e^{(-d_i + C_1 \varepsilon_1)s' - j}. \end{aligned}$$

By summing over all $y_1 \in \mathfrak{J} \cap \Sigma^r$ with

$$\Omega'(y_1) \cap \mathfrak{Z}_j'' \neq \emptyset,$$

we get

$$\mathcal{Z}_j(\Sigma^r) \geq \beta^{52} \tilde{\mathfrak{m}} \beta^{53} e^{(-d_i + C_1 \varepsilon_1)s' - j} \geq \tilde{\mathfrak{m}} e^{(-d_i + C_1 \varepsilon_1 - \varepsilon_1)s' - j}.$$

For upper bound, for $y_1 \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I} \cap \Sigma^r$ such that $\Theta'(z)$ contains a

$$\Xi_j(z') \subset \mathfrak{Z}'_j,$$

if there exists $y_2 \in \mathcal{I} \cap \Xi_j(z')$ with $(y_1, y_2) \in \diamond'_{r,j}$, we will get

$$\Xi_j(z') \subset \Sigma_{s',j-2}(y_1).$$

Let us define

$$(10.4) \quad \begin{aligned} \mathcal{I}'(y_1, j) &:= \left\{ z \in \mathcal{I} \cap \Sigma^r \cap \mathfrak{Z}'_j : \text{the heavy slice } \Xi_j(z') \subset \Theta'(z) \right. \\ &\quad \left. \text{satisfies } \Xi_j(z') \subset \Sigma_{s',j-2}(y_1) \cap \mathfrak{Z}'_j \right\}. \end{aligned}$$

For $z \in \mathcal{I}'(y_1, j)$, let us define

$$\mathfrak{J}(y_1, z, j) := \{y_2 \in \mathcal{I}_z \cap \Sigma_{s',j-2}(y_1) \cap \mathfrak{Z}'_j : (y_1, y_2) \in \diamond'_{r,j}\}.$$

Then, by [Lemma 8.15](#), for every $z \in \mathcal{I}'(y_1, j)$ and for any $y_2 \in \mathfrak{J}(y_1, z, j)$, we have

$$\mathfrak{D}_z(y_2) \in \mathcal{C}(\mathcal{T}(y_1), z) \cap \mathfrak{B}(\mathcal{T}(y_1), z).$$

Therefore,

$$\sum_{y_2 \in \mathfrak{J}(y_1, z, j)} \mathcal{W}(y_1) \mathcal{W}(y_2) \leq \mathcal{W}(y_1) \beta^{-198} e^{-j} e^{-2d_i s'}.$$

Moreover, since $\tilde{\mu}(\Sigma_{s',j-2}(y_1)) \leq e^{-2s'}e^{-d_0(j-2)}$, we have

$$\begin{aligned} |\mathcal{I}'(y_1, j)| &\leq e^{-2s'}e^{-d_0(j-2)}(1-\beta)^{-j}e^{2s'}e^{(d_i-C_1\varepsilon_1)s'} \\ &\leq e^{(d_i-C_1\varepsilon_1+\varepsilon_1)s'-d_0j}. \end{aligned}$$

By summing up all $y_1 \in \mathfrak{J} \cap \Sigma^r$ and $z \in \mathcal{I}'(y_1, j)$, we get

$$\begin{aligned} \mathcal{Z}_j(\Sigma^r) &\leq \sum_{y_1 \in \mathfrak{J} \cap \Sigma^r} \mathcal{W}(y_1) |\mathcal{I}'(y_1, j)| \beta^{-198} e^{-j} e^{-2d_i s'} \\ &\leq \sum_{y_1 \in \mathfrak{J} \cap \Sigma^r} \mathcal{W}(y_1) e^{(d_i-C_1\varepsilon_1+\varepsilon_1)s'-d_0j} \beta^{-198} e^{-j} e^{-2d_i s'} \\ &\leq \tilde{\mathfrak{m}} e^{(-d_i-C_1\varepsilon_1+2\varepsilon_1)s'-j-d_0j}. \end{aligned}$$

The upper bound is smaller than the lower bound, leading to a contradiction. This proves that

$$\tilde{\mu}(\mathfrak{Z}'_j) \leq \beta^{50.1}. \quad \square$$

11. Dimension improvement — structured part

Let us keep notation as in [Definition 9.8](#). This section is devoted to the study of the structured part $\bar{\mathcal{R}}_i^s$.

We will prove the following statement:

PROPOSITION 11.1. *Suppose $d_i \leq 4 - 10^{11}C_1\varepsilon_1$. We have*

$$\tilde{\mu}(\bar{\mathcal{R}}_i^s) \leq \beta^{22}.$$

Note that by [equations \(9.3\) and \(9.9\)](#), [Propositions 9.9, 9.10](#) and [11.1](#) will imply [inequality \(9.1\)](#) and thus [Proposition 9.1](#).

Note that up to now, from the results of [Subsection 6.2](#) we have only used [Proposition 6.2](#). [Proposition 11.1](#) uses [Proposition 6.3](#) in an essential way, which basically says that $\tilde{\mu}$ cannot concentrate along $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ -orbits.

Let us first fix some notation.

Definition 11.2. For $z \in \mathfrak{J}$, if

$$(11.1) \quad \tilde{\mu}(\Omega'(z)) \geq e^{-2s'}e^{-2d_i s'-8C_1\varepsilon_1 s'},$$

we will call $\Omega'(z)$ *full*. Otherwise, we call it *almost empty*.

Definition 11.3. For $z \in \mathcal{I}$, we say that two curve neighborhoods $\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z)$ and $\mathcal{C}' = \mathcal{C}(\mathcal{T}(y'), z)$ are *comparable* if $\mathcal{C}'$ is contained in a $e^{-3s'+11C_1\varepsilon_1 s'}$ neighborhood of $\mathcal{C}$. Let us write

$$\mathcal{C}' \approx \mathcal{C},$$

if $\mathcal{C}$ and $\mathcal{C}'$ are comparable.

We need the following lemma:

LEMMA 11.4. Let $C \subset \mathfrak{C} = [1.2\beta e^{-s'}]^2$ be a hyperbola with the following equation:

$$X = \frac{-Y}{w_1 Y + k},$$

where $|w_1| \in [e^{s'-10C_1\varepsilon_1 s'}, \beta e^{s'}]$ and $|k| \in [e^\ell, e^{\ell+1}]$ for some $\ell \in \mathbb{Z}_{\geq 0}, \ell \leq s'$. Let $C' \subset \mathfrak{C}$ be another hyperbola with the following equation:

$$X = \frac{-Y}{w'_1 Y + k'},$$

where $|w'_1| \in [e^{s'-10C_1\varepsilon_1 s'}, \beta e^{s'}]$.

We will prove the following two parts:

- (1) For $0 < A < e^{2s'-C_1\varepsilon_1 s'}$, if C' is contained in the $Ae^{-3s'}$ neighborhood of C , then we have

$$(11.2) \quad |w_1 - w'_1| \leq Ae^{-s'+2\ell}, \text{ and } |k - k'| \leq Ae^{-2s'+2\ell}.$$

Conversely, if (11.2) holds, then C' is contained in the $Ae^{-3s'}$ neighborhood of C .

- (2) Suppose C' is not contained in the $e^{-3s'+11C_1\varepsilon_1 s'}$ neighborhood of C . For $0 < A < e^{2s'-C_1\varepsilon_1 s'}$, if we have

$$(11.3) \quad |w_1 - w'_1| \in [Ae^{-s'+2\ell}, Ae^{-s'+2\ell+1}]$$

and there exists $\mathbf{p}' \in C'$ such that $\|\mathbf{p}'\| \geq e^{-s'-10C_1\varepsilon_1 s'}$ and $\mathbf{p}'$ is contained in the $\beta^{-100}e^{-3s'}$ neighborhood of C , then C' is contained in the $\beta^{-100}Ae^{-3s'}$ neighborhood of C . Moreover, if $\mathbf{p}'_1, \mathbf{p}'_2 \in C'$ satisfy $\|\mathbf{p}'_1\|, \|\mathbf{p}'_2\| \geq e^{-s'-10C_1\varepsilon_1 s'}$ and $\mathbf{p}'_1, \mathbf{p}'_2$ are both contained in the $\beta^{-100}e^{-3s'}$ neighborhood of C , then

$$\|\mathbf{p}'_1 - \mathbf{p}'_2\| \leq A^{-1}e^{-s'+11C_1\varepsilon_1 s'}.$$

Proof. Let us first prove Part 1. Our assumption is that for all $Y \in [\beta e^{-s'}]$,

$$\left| \frac{Y}{w_1 Y + k} - \frac{Y}{w'_1 Y + k'} \right| \leq Ae^{-3s'}.$$

This implies that

$$|Y| \cdot |w_1 - w'_1| \frac{\left| Y + \frac{k'-k}{w_1-w'_1} \right|}{|w_1 Y + k| \cdot |w'_1 Y + k'|} \leq Ae^{-3s'}.$$

Then it is easy to get that $|k'| \in [e^{\ell-1}, e^{\ell+2}]$.

Let us consider the following two subcases:

- (1) $|k - k'| < e^{-s'}|w_1 - w'_1|$;
(2) $|k - k'| \geq e^{-s'}|w_1 - w'_1|$.

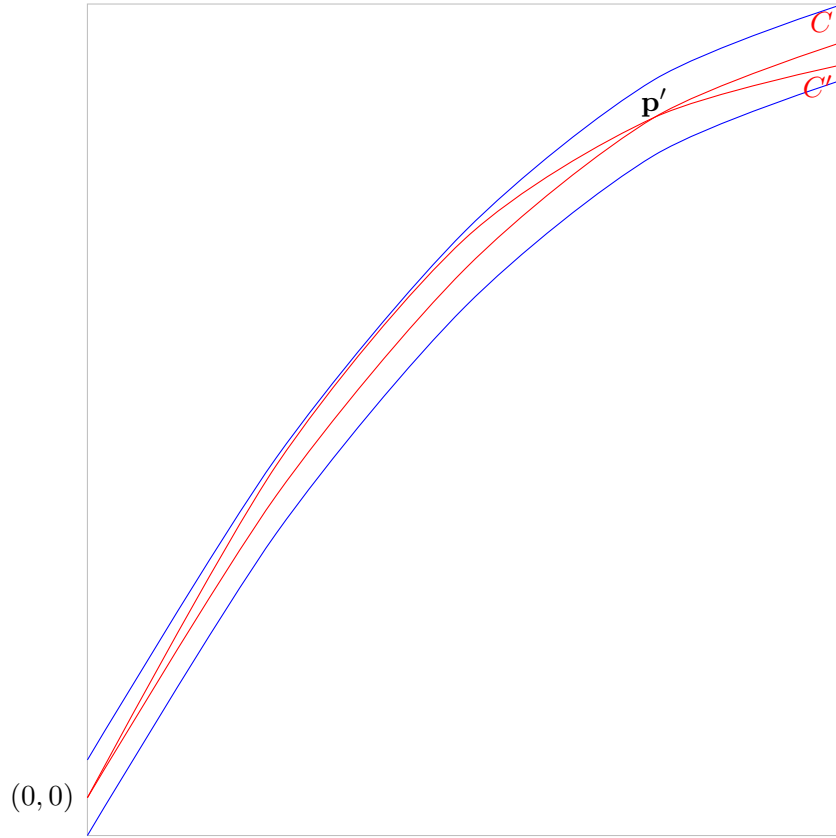


Figure 1. [Lemma 11.4](#), part 2 — if there is a $\mathbf{p}' \in C'$, not too close to the origin, near C , then all of C' is within a comparable distance to C .

Subcase 1. $|k - k'| < e^{-s'}|w_1 - w'_1|$; we have

$$|w_1 - w'_1|e^{-2s'-2\ell} \leq Ae^{-3s'},$$

hence

$$|w_1 - w'_1| \leq Ae^{-s'+2\ell}.$$

Moreover

$$|k - k'| < e^{-s'}|w_1 - w'_1| \leq Ae^{-2s'+2\ell},$$

establishing [\(11.2\)](#).

Subcase 2. $|k - k'| \geq e^{-s'}|w_1 - w'_1|$; we have

$$|k - k'|e^{-s'-2\ell} \leq Ae^{-3s'},$$

implying

$$|k - k'| \leq Ae^{-2s'+2\ell}.$$

Then

$$|w_1 - w'_1| \leq e^{s'} |k - k'| \leq Ae^{-s'+2\ell}.$$

This proves (11.2) and hence the first statement of Part 1.

Now let us assume (11.2); we want to prove that for any $Y \in [\beta e^{-s'}]$,

$$\left| \frac{Y}{w_1 Y + k} - \frac{Y}{w'_1 Y + k'} \right| \leq Ae^{-3s'}.$$

This is equivalent to

$$|Y| \cdot |w_1 - w'_1| \frac{\left| Y + \frac{k' - k}{w_1 - w'_1} \right|}{|w_1 Y + k| \cdot |w'_1 Y + k'|} \leq Ae^{-3s'}.$$

Note that the left-hand side is bounded from above by

$$(11.4) \quad e^{-s'-2\ell} |w_1 - w'_1| \cdot \left| Y + \frac{k' - k}{w_1 - w'_1} \right|.$$

Let us again consider the above two subcases.

For Subcase 1, we have

$$\begin{aligned} |w_1 - w'_1| \cdot \left| Y + \frac{k' - k}{w_1 - w'_1} \right| &\leq e^{-s'} |w_1 - w'_1| \\ &\leq e^{-s'} Ae^{-s'+2\ell} = Ae^{-2s'+2\ell}. \end{aligned}$$

For Subcase 2, we have

$$|w_1 - w'_1| \cdot \left| Y + \frac{k' - k}{w_1 - w'_1} \right| \leq |k - k'| \leq Ae^{-2s'+2\ell}.$$

This shows that for both subcases we have that (11.4) is bounded from above by $Ae^{-3s'}$, which implies the second statement of Part 1.

Let us prove Part 2. By the assumptions, we have that there exists

$$\tilde{Y} \in [e^{-s'-10C_1\varepsilon_1 s'}, \beta e^{-s'}]$$

such that

$$\left| \frac{\tilde{Y}}{w_1 \tilde{Y} + k} - \frac{\tilde{Y}}{w'_1 \tilde{Y} + k'} \right| \leq \beta^{-100} e^{-3s'},$$

which implies

$$|\tilde{Y}| \cdot |w_1 - w'_1| \frac{\left| \tilde{Y} + \frac{k' - k}{w_1 - w'_1} \right|}{|w_1 \tilde{Y} + k| \cdot |w'_1 \tilde{Y} + k'|} \leq \beta^{-100} e^{-3s'}.$$

It is easy to see that $|k'| \in [e^{\ell-1}, e^{\ell+1}]$.

We claim that

$$(11.5) \quad \left| \frac{k' - k}{w'_1 - w_1} \right| \leq e^{-s'}.$$

In fact, if this is not the case, we have,

$$\left| \frac{k' - k}{w'_1 - w_1} \right| \ll \left| \tilde{Y} + \frac{k' - k}{w'_1 - w_1} \right|,$$

which implies that

$$\frac{|\tilde{Y}| \cdot |k' - k|}{|w_1 \tilde{Y} + k| \cdot |w'_1 \tilde{Y} + k'|} \leq \beta^{-100} e^{-3s'}.$$

Thus we have

$$|k' - k| \leq e^{-2s' + 2\ell + 11C_1\varepsilon_1 s'}.$$

Then for any $Y' \in [\beta e^{-s'}]$, we have

$$\begin{aligned} \left| \frac{Y'}{w_1 Y' + k} - \frac{Y'}{w'_1 Y' + k'} \right| &\leq \frac{|Y'| \cdot |k' - k|}{|w_1 Y' + k| \cdot |w'_1 Y' + k'|} \\ &\leq e^{-s'} e^{-2s' + 2\ell + 11C_1\varepsilon_1 s'} e^{-2\ell} = e^{-3s' + 11C_1\varepsilon_1 s'}. \end{aligned}$$

This implies that C' is contained in the $e^{-3s' + 11C_1\varepsilon_1 s'}$ neighborhood of C , which contradicts our assumption. This proves the claim (11.5).

From (11.3) and (11.5), we obtain that, for any $Y \in [\beta e^{-s'}]$,

$$|Y| \cdot |w_1 - w'_1| \frac{\left| Y + \frac{k' - k}{w_1 - w'_1} \right|}{|w_1 Y + k| \cdot |w'_1 Y + k'|} \leq A \beta^{-100} e^{-3s'}.$$

This proves that C' is in the $\beta^{-100} A e^{-3s'}$ neighborhood of C .

Finally, if $\tilde{Y}' \in [e^{-s' - 10C_1\varepsilon_1 s'}, \beta e^{-s'}]$ also satisfies

$$\left| \frac{\tilde{Y}'}{w_1 \tilde{Y}' + k} - \frac{\tilde{Y}'}{w'_1 \tilde{Y}' + k'} \right| \leq \beta^{-100} e^{-3s'},$$

then we have

$$\left| \tilde{Y} + \frac{k' - k}{w_1 - w'_1} \right| \leq A^{-1} \beta^{-100} e^{-s' + 10C_1\varepsilon_1 s'},$$

and

$$\left| \tilde{Y}' + \frac{k' - k}{w_1 - w'_1} \right| \leq A^{-1} \beta^{-100} e^{-s' + 10C_1\varepsilon_1 s'}.$$

This implies that

$$|\tilde{Y} - \tilde{Y}'| \leq A^{-1} e^{-s' + 11C_1\varepsilon_1 s'}.$$

This completes the proof. $\square$

Let us apply Lemma 11.4 to prove the following lemmata:

LEMMA 11.5. *For $y \in \mathfrak{J} \cap \Sigma^r$, $z \in \mathcal{I} \cap \Sigma^r$, let us denote*

$$\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z) \subset \mathfrak{C} = [1.2\beta e^{-s'}]^2.$$

Suppose that $\mathcal{C}$ has equation

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0$$

with $\|(v_1, w_1)\| \geq e^{s'-10C_1\varepsilon_1s'}$ and

$$\max\{|v_1/w_1|, |w_1/v_1|\} \in [e^\ell, e^{\ell+1}].$$

Then, for $\ell < s'$, there exist at most $e^{2s'+2\ell+60C_1\varepsilon_1s'}$ different $y' \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C}(\mathcal{T}(y'), z) \approx \mathcal{C}.$$

For $\ell \geq s'$, there are at most $e^{4s'+60C_1\varepsilon_1s'}$ different $y' \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C}(\mathcal{T}(y'), z) \approx \mathcal{C}.$$

Proof. We first consider the case $\ell < s'$. For $A > 1$, let us denote by $\mathcal{C}[A]$ the $Ae^{-3s'}$ neighborhood of $\mathcal{C}$.

Let us fix $z_0 \in \mathcal{I}_z$ such that $\mathfrak{D}_z(z_0) = (X'_0, Y'_0) \in \mathcal{C}[e^{11C_1\varepsilon_1s'}]$, and let us consider all $y' \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C} \approx \mathcal{C}(\mathcal{T}(y'), z).$$

Assume in addition that y' is such that there are coefficients $v_{1,y'}, w_{1,y'}$ of size not exceeding $\beta e^{s'}$ and a point $y'_0 \in \Omega(y')$ such that

$$z_0 = \exp(v_{1,y'}\mathbf{v}_1 + w_{1,y'}\mathbf{w}_1)u(r_{y'})y'_0.$$

Let us denote by $\mathcal{M}(\mathcal{C}, z_0)$ the collection of those y' 's.

Fix a $y \in \mathcal{M}(\mathcal{C}, z_0)$ and consider the curve neighborhood $\mathcal{C}'$ with equation

$$(X - X'_0)(Y - Y'_0) + v_{1,y}^{-1}(X - X'_0) + w_{1,y}^{-1}(Y - Y'_0) = 0.$$

Then it is easy to see that for any $y' \in \mathcal{M}(\mathcal{C}, z_0)$,

$$\mathcal{C}(\mathcal{T}(y'), z) \subset \mathcal{C}'[2e^{11C_1\varepsilon_1s'}].$$

Without loss of generality, let us assume that $(X'_0, Y'_0) = \mathbf{0}$. Then the equation reads

$$XY + v_{1,y}^{-1}X + w_{1,y}^{-1}Y = 0.$$

Then for any $y' \in \mathcal{M}(\mathcal{C}, z_0)$, if y'_0 is the corresponding point satisfying

$$z_0 = \exp(v_{1,y'}\mathbf{v}_1 + w_{1,y'}\mathbf{w}_1)u(r_{y'})y'_0,$$

we get that the curve

$$(11.6) \quad XY + v_{1,y'}^{-1}X + w_{1,y'}^{-1}Y = 0$$

is contained in $2e^{-3s'+10C_1\varepsilon_1s'}$ neighborhood of the curve

$$(11.7) \quad XY + v_{1,y}^{-1}X + w_{1,y}^{-1}Y = 0.$$

Let us assume that $|w_{1,y}| \geq |v_{1,y}|$. The opposite case can be handled similarly.

Let us denote

$$k = \frac{w_{1,y}}{v_{1,y}} \quad \text{and} \quad k' = \frac{w_{1,y'}}{v_{1,y'}}.$$

Then it is easy to see that $k \in [e^\ell, e^{\ell+1}]$ and $k' \in [e^{\ell-1}, e^{\ell+2}]$. Note that the equations of the curves (11.6) and (11.7) can be written as

$$X = \frac{-Y}{w_{1,y}Y + k} \quad \text{and} \quad X = \frac{-Y}{w_{1,y'}Y + k'},$$

respectively. Then for all $Y \in [\beta e^{-s'}]$, we have

$$\left| \frac{Y}{w_{1,y}Y + k} - \frac{Y}{w_{1,y'}Y + k'} \right| \leq 2e^{-3s'+11C_1\varepsilon_1s'}.$$

By Lemma 11.4 Part 1 with $A = 2e^{11C_1\varepsilon_1s'}$, we have that

$$|w_{1,y} - w_{1,y'}| \leq 2e^{-s'+11C_1\varepsilon_1s'+2\ell} \quad \text{and} \quad |k - k'| \leq 2e^{-2s'+11C_1\varepsilon_1s'+2\ell}.$$

Then there are at most $e^{40C_1\varepsilon_1s'+2\ell}$ different $y' \in \mathfrak{J} \cap \Sigma^r$ contained in $\mathcal{M}(\mathcal{C}, z_0)$. By running over all $z_0 \in \mathcal{I}_z$ such that $\mathfrak{D}_z(z_0) \in \mathcal{C}[e^{11C_1\varepsilon_1s'}]$, we have that there are at most

$$e^{2s'+2\ell+60C_1\varepsilon_1s'}$$

different $y' \in \mathfrak{J} \cap \Sigma^r$ such that $\mathcal{C}(\mathcal{T}(y'), z) \approx \mathcal{C}$. This proves the statement for $\ell < s'$.

Now let us turn to the case $\ell \geq s'$. In this case, by Remark 8.16, we can consider the corresponding straight lines

$$X = -Y/k, \text{ and } X = -Y/k'.$$

Then, for any $y \in [\beta e^{-s'}]$, we have

$$|Y/k - Y/k'| \leq 2e^{-3s'+11C_1\varepsilon_1s'},$$

which implies that

$$|k - k'| \cdot |kk'|^{-1} \leq 2e^{-2s'+11C_1\varepsilon_1s'}.$$

This implies that for any fixed $w_{1,y'}$, the desired $v_{1,y'}$ is lying in

$$\frac{w_{1,y'}}{k} + [2e^{-s'+11C_1\varepsilon_1s'}].$$

Noting that $w_{1,y'} \in [\beta e^{s'}]$, we have that there are at most $e^{40C_1\varepsilon_1s'+2s'}$ different $y' \in \mathfrak{J} \cap \Sigma^r$ contained in $\mathcal{M}(\mathcal{C}, z_0)$. By running over all $z_0 \in \mathcal{I}_z$ such that $\mathfrak{D}_z(z_0) \in \mathcal{C}[e^{11C_1\varepsilon_1s'}]$, we have that there are at most

$$e^{4s'+60C_1\varepsilon_1s'}$$

different $y' \in \mathfrak{J} \cap \Sigma^r$ such that $\mathcal{C}(\mathcal{T}(y'), z) = \mathcal{C}$.

This completes the proof. $\square$

LEMMA 11.6. *Let us fix constants $\ell, m \geq 1$. Let $\mathcal{C} \subset \mathfrak{C}$ be the $\beta^{-100}e^{-3s'}$ -neighborhood of the curve*

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0$$

with $\|(v_1, w_1)\| \geq e^{s'-10C_1\varepsilon_1s'}$ and

$$\max\{|v_1/w_1|, |w_1/v_1|\} \in [e^\ell, e^{\ell+1}].$$

Let $\mathcal{C}'$ be the $\beta^{-100}e^{-3s'}$ -neighborhood of the curve with the following equation

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0.$$

Suppose we have that $\mathcal{C}$ and $\mathcal{C}'$ are not comparable. Moreover, suppose we have the following:

$$\max\{|v'_1/w'_1|, |w'_1/v'_1|\} \in [e^\ell, e^{\ell+1}],$$

$$\max\{|v'_1 - v_1|, |w'_1 - w_1|\} \in [e^{s'-m}, e^{s'-m+1}],$$

and suppose that there exists $(X_1, Y_1) \in \mathcal{C} \cap \mathcal{C}'$ with

$$\max\{|X_0 - X_1|, |Y_0 - Y_1|\} \geq e^{-s'-10C_1\varepsilon_1s'}.$$

Then $m+2\ell \leq 2s'-10C_1\varepsilon_1s'$, and $\mathcal{C}'$ is contained in the $e^{-s'-m-2\ell}$ neighborhood of $\mathcal{C}$.

Proof. Without loss of generality, we can assume that $(X_0, Y_0) = \mathbf{0}$ and $|w_1/v_1| \in [e^\ell, e^{\ell+1}]$. Then we have $|w'_1/v'_1| \in [e^\ell, e^{\ell+1}]$ and $|w_1 - w'_1| \in [e^{s'-m}, e^{s'-m+1}]$, and there exists $(X_1, Y_1) \in \mathcal{C} \cap \mathcal{C}'$ with $|Y_1| \geq e^{-s'-10C_1\varepsilon_1s'}$.

Let us rewrite the equation of $\mathcal{C}$ as

$$X = \frac{-Y}{w_1Y + k}, \text{ where } k = w_1/v_1$$

and the equation of $\mathcal{C}'$ as

$$X = \frac{-Y}{w'_1Y + k'}, \text{ where } k' = w'_1/v'_1.$$

Since $(X_1, Y_1) \in \mathcal{C} \cap \mathcal{C}'$, we have

$$\left| \frac{Y_1}{w_1Y_1 + k} - \frac{Y_1}{w'_1Y_1 + k'} \right| \leq \beta^{-100}e^{-3s'}.$$

By [Lemma 11.4 Part 2](#) with $A = e^{2s'-2\ell-m}$, we have that $\mathcal{C}'$ is contained in the $e^{-s'-m-2\ell}$ -neighborhood of $\mathcal{C}$. Recalling that by assumption $\mathcal{C}'$ and $\mathcal{C}$ are not comparable, we conclude that

$$m + 2\ell \leq 2s' - 10C_1\varepsilon_1s'. \quad \square$$

LEMMA 11.7. *Let us fix constants $m, \ell \geq 1$ with $m + 2\ell \leq 2s' - 10C_1\varepsilon_1s'$. Let $\mathcal{C} \subset \mathfrak{C}$ be the $\beta^{-100}e^{-3s'}$ -neighborhood of the curve*

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0,$$

with $\|(v_1, w_1)\| \geq e^{s'-10C_1\varepsilon_1s'}$ and

$$\max\{|v_1/w_1|, |w_1/v_1|\} \in [e^\ell, e^{\ell+1}].$$

Let $\mathfrak{N}(\mathcal{C}, (X_0, Y_0), m)$ denote a collection of non-comparable $\beta^{-100}e^{-3s'}$ -curve neighborhoods with equations of the form

$$(X - X_0)(Y - Y_0) + v_1'^{-1}(X - X_0) + w_1'^{-1}(Y - Y_0) = 0$$

and contained in the $e^{-s'-m-2\ell}$ neighborhood of $\mathcal{C}$. Then

$$|\mathfrak{N}(\mathcal{C}, (X_0, Y_0), m)| \leq e^{4s'-2m-4\ell+100C_1\varepsilon_1s'}.$$

Moreover, for $\mathcal{C}' \in \mathfrak{N}(\mathcal{C}, (X_0, Y_0), m)$ with corresponding parameters v_1', w_1' , let m' denote the number satisfying

$$\max\{|w_1' - w_1|, |v_1' - v_1|\} \in [e^{s'-m'}, e^{s'-m'+1}].$$

Then the intersection

$$\mathcal{C} \cap \mathcal{C}' \cap \{(X, Y) : \|(X, Y) - (X_0, Y_0)\| \geq e^{-s'-10C_1\varepsilon_1s'}\}$$

contains at most $e^{m'+2\ell+11C_1\varepsilon_1s'}$ different points from the set

$$\{\mathfrak{D}_z(z') : z' \in \mathcal{I}_z\}.$$

Proof. Without loss of generality, we can assume that $(X_0, Y_0) = \mathbf{0}$, and $|w_1| \geq |v_1|$.

Let us write the equation of $\mathcal{C}$ as

$$X = \frac{-Y}{w_1Y + k}, \text{ where } k = w_1/v_1.$$

Then $|k| \in [e^\ell, e^{\ell+1}]$. Suppose $\mathcal{C}'$ with equation

$$X = \frac{-Y}{w_1'Y + k'} \text{ where } k' = w_1'/v_1'$$

is in $\mathfrak{N}(\mathcal{C}, \mathbf{0}, m)$. Then, for all $Y \in [\beta e^{-s'}]$, we have

$$\left| \frac{Y}{w_1Y + k} - \frac{Y}{w_1'Y + k'} \right| \leq e^{-s'-m-2\ell}.$$

By [Lemma 11.4 Part 1](#) with $A = e^{2s'-m-2\ell}$, we have

$$(11.8) \quad |w_1 - w_1'| \leq e^{s'-m}, \text{ and } |k - k'| \leq e^{-m}.$$

Note that by [Lemma 11.4 Part 1](#) with $A = e^{11C_1\varepsilon_1s'}$, if

$$|w_1' - \bar{w}_1'| \leq e^{-s'+11C_1\varepsilon_1s'+2\ell} \quad \text{and} \quad |k' - \bar{k}'| \leq e^{-2s'+11C_1\varepsilon_1s'+2\ell},$$

the corresponding curves

$$X = \frac{-Y}{w_1'Y + k'} \quad \text{and} \quad X = \frac{-Y}{\bar{w}_1'Y + \bar{k}'}$$

are comparable. Thus, the [conditions \(11.8\)](#) give at most

$$e^{4s'-2m-4\ell}$$

different curve neighborhoods that are not comparable to each other.

Let us prove the last statement. By [Lemma 11.4](#) (2) with $A = e^{2s'-2\ell-m'}$, we have that there exists $(X_1, Y_1) \in \mathcal{C}$ such that the intersection

$$\mathcal{C} \cap \mathcal{C}' \cap \left\{ (X, Y) : \|(X, Y) - (X_0, Y_0)\| \geq e^{-s'-10C_1\varepsilon_1 s'} \right\}$$

is contained in the $e^{-3s'+m'+2\ell+11C_1\varepsilon_1 s'}$ -neighborhood of (X_1, Y_1) , denoted by $B(X_1, Y_1)$. Then the statement follows from the fact that the intersection

$$\mathcal{C} \cap B(X_1, Y_1)$$

contains at most $e^{m'+2\ell+11C_1\varepsilon_1 s'}$ different elements from

$$\{\mathfrak{D}_z(z') : z' \in \mathcal{I}_z\}.$$

This completes the proof. $\square$

We also need the following lemma:

LEMMA 11.8. *Given $(X_0, Y_0) \in \mathfrak{C} = [1.2\beta e^{-s'}]^2$, and $1 < \ell \leq s'$, let us denote*

$$\mathcal{N}^\ell = \mathcal{N}^\ell(X_0, Y_0) := (X_0, Y_0) + [\beta e^{-s'-\ell}] \times [\beta e^{-s'}].$$

Suppose we have a cross-section Σ^r and $z \in \mathcal{I} \cap \Sigma^r$, $y \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z) \subset \mathcal{N}^\ell.$$

Then there exists $z' \in \Theta(z)$ only depending on $\mathcal{N}^\ell$ (and not on $\mathcal{C}$) such that $\mathfrak{D}_z(z') \in \mathcal{N}^\ell$ and

$$\Omega'(y) \subset a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)a(-2s')z'.$$

See, for example, [equation \(6.1\)](#) for the definition of $Q(\mathfrak{r}_1, \ell, \beta)$.

Proof. First, we note that there are $\tilde{y} \in \Omega(y)$ and $\tilde{z} \in \Theta(z)$ such that

$$(11.9) \quad \tilde{y} = u(r) \exp(\mathbf{w}^+) \tilde{z}$$

and

$$\tilde{z} = \exp(\mathbf{w}^-)z,$$

where $\mathbf{w}^+ = v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1 \in B_{\mathbf{r}^+}(\beta e^{s'})$, $r \in [\beta e^{2s'}]$ and $\mathbf{w}^- = X_1 \mathbf{v}_2 + Y_1 \mathbf{w}_2 \in \mathbf{r}^-$. By [Lemma 8.15](#), we have that $\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z)$ is the $\beta^{-100}e^{-3s'}$ -neighborhood of the following curve (v_1, w_1) as above):

$$(X - X_1)(Y - Y_1) + v_1^{-1}(X - X_1) + w_1^{-1}(Y - Y_1) = 0.$$

Since $\mathcal{C} \subset \mathcal{N}^\ell$, we have that $(X_1, Y_1) \in \mathcal{N}^\ell$. Moreover, for any $Y \in [\beta e^{-s'}]$, we have

$$|X - X_1| = \frac{|Y - Y_1|}{|w_1(Y - Y_1) + k|} \leq \beta e^{-s'-\ell},$$

where $k = w_1/v_1$. Note that there exists $Y \in [\beta e^{-s'}]$ such that $|Y - Y_1| \geq \beta e^{-s'}$; we get

$$|w_1(Y - Y_1) + k| \geq e^\ell.$$

The above bounds on w_1 and $|Y - Y_1|$ imply that $|w_1(Y - Y_1)| \leq \beta^2$, hence

$$|k| = |w_1/v_1| \geq e^\ell.$$

Therefore, we have

$$(11.10) \quad |v_1| \leq e^{-\ell} |w_1| \leq \beta e^{s'-\ell}.$$

Let $z' \in \Theta(z)$ be the following element:

$$z' = \exp(\mathbf{w}_0^-)z,$$

where $\mathbf{w}_0^- = X_0 \mathbf{v}_2 + Y_0 \mathbf{w}_2$. Then we have that

$$(11.11) \quad \tilde{z} = \exp(\tilde{\mathbf{w}}^-) u^*(r^*) z',$$

where $|r^*| \leq \beta e^{-2s'}$ and $\tilde{\mathbf{w}}^- = \tilde{X} \mathbf{v}_2 + \tilde{Y} \mathbf{w}_2$ with $(\tilde{X}, \tilde{Y}) \in \mathcal{N}^\ell(\mathbf{0})$.

By (11.9) and (11.11), we get

$$\tilde{y} = u(r) \exp(\mathbf{w}^+) \exp(\tilde{\mathbf{w}}^-) u^*(r^*) z',$$

where $\mathbf{w}^+ \in B_{\mathbf{r}^+}(\beta e^{s'})$ with $\|p_{\mathbf{v}_1}(\mathbf{w}^+)\| \leq \beta e^{s'-\ell}$ by (11.10), and $\tilde{\mathbf{w}}^- \in B_{\mathbf{r}^-}(\beta e^{-s'})$ with $\|p_{\mathbf{v}_2}(\tilde{\mathbf{w}}^-)\| \leq \beta e^{-s'-\ell}$ since $(\tilde{X}, \tilde{Y}) \in \mathcal{N}^\ell(\mathbf{0})$. Then we can write

$$\tilde{y} = a(2s') u(\bar{r}) \exp(\bar{\mathbf{w}}^+) \exp(\bar{\mathbf{w}}^-) u^*(\bar{r}^*) a(-2s') z',$$

where $|\bar{r}|, |\bar{r}^*| \leq \beta$, $\bar{\mathbf{w}}^+ \in B_{\mathbf{r}^+}(\beta)$ with $\|p_{\mathbf{v}_1}(\bar{\mathbf{w}}^+)\| \leq \beta e^{-\ell}$, and $\bar{\mathbf{w}}^- \in B_{\mathbf{r}^-}(\beta)$ with $\|p_{\mathbf{v}_2}(\bar{\mathbf{w}}^-)\| \leq \beta e^{-\ell}$. Note that these estimates imply that

$$\|[\bar{\mathbf{w}}^-, \bar{\mathbf{w}}^+]\| \leq \beta^2 e^{-\ell}.$$

By Lemma 2.3, we have

$$\exp(\mathbf{w}^+) \exp(\tilde{\mathbf{w}}^-) = \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}),$$

where $\bar{\mathbf{w}} \in B_{\mathbf{r}_1+\mathbf{r}_2}(\beta)$ with $\|p_{\mathbf{r}_1}(\bar{\mathbf{w}})\| \leq \beta e^{-\ell}$ and $\bar{\mathbf{h}} \in B_{\mathbf{h}+\mathbb{R}\mathbf{a}_0}(\beta e^{-\ell})$. Thus, we have

$$\begin{aligned} \tilde{y} &= a(2s') u(\bar{r}) \exp(\bar{\mathbf{w}}) \exp(\bar{\mathbf{h}}) u^*(\bar{r}^*) a(-2s') z' \\ &\in a(2s') Q(\mathbf{r}_1, \ell, \beta) B^{A_0 H}(\beta) a(-2s') z', \end{aligned}$$

which implies that

$$\Omega'(y) \subset a(2s') Q(\mathbf{r}_1, \ell - 1, \beta) B^{A_0 H}(\beta) a(-2s') z'. \quad \square$$

Remark 11.9. Lemma 11.8 roughly says that given $z \in \mathcal{I} \cap \Sigma^r$ and

$$\mathcal{N}^\ell = \mathcal{N}^\ell(X_0, Y_0) \subset \mathfrak{C}$$

as above, all $y \in \mathfrak{J} \cap \Sigma^r$ satisfying

$$\mathcal{C}(\mathcal{T}(y), z) \subset \mathcal{N}^\ell$$

are contained in a $a(2s')$ -translate of a $\beta e^{-\ell+1}$ -neighborhood of a piece of some $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{R}^2$ -orbit of size β .

Now we are ready to prove [Proposition 11.1](#)

Proof of Proposition 11.1. Assume in contradiction that

$$\tilde{\mu}(\bar{\mathcal{R}}_i^s) > \beta^{22}.$$

Let $\mathfrak{Z} \subset \Pi(x)$ be the subset from [Proposition 10.1](#). By replacing $\bar{\mathcal{R}}_i^s$ with $\bar{\mathcal{R}}_i^s \setminus \mathfrak{Z}$, we can assume that (10.1) holds for any $z \in \mathcal{I} \cap \bar{\mathcal{R}}_i^s$.

Thus we may assume:

$$(11.12) \quad \begin{aligned} & \mathcal{I} \cap \bar{\mathcal{R}}_i^s \neq \emptyset, \\ & \tilde{\mu}(\Theta'(z)) \leq e^{-2s'} e^{-(d_i - C_1 \varepsilon_1)s'} \quad \text{for any } z \in \mathcal{I} \cap \bar{\mathcal{R}}_i^s \end{aligned}$$

Let us fix a $z \in \mathcal{I} \cap \bar{\mathcal{R}}_i^s$. By the definition of $\bar{\mathcal{R}}_i^s$ (see [Definition 9.8](#)), we have that $\Theta'(z)$ is highly structured. Thus, there exists $r = r(z)$ such that $z \in \Sigma^r$ and

$$(11.13) \quad \tilde{\mu}(\mathcal{SS}(z, r)) \geq e^{-2s' - 10C_1 \varepsilon_1 s'}$$

Recall that by (9.2), for any y ,

$$\tilde{\mu}(\Omega'(y)) \leq \beta^{-15} e^{-2s'} e^{-2d_i s'}.$$

Recall also that $\mathcal{SS}(z, r)$ is the union of $\Omega'(y)$ with $y \in \mathcal{HS}(z, r)$, i.e., those $y \in \mathfrak{J} \cap \Sigma^r$ for which $\mathcal{C}(\mathcal{T}(y), z)$ is highly concentrated. Then by (11.13) and (9.2), we have the following:

Assumption 11.10. There exist at least $e^{2d_i s' - 15C_1 \varepsilon_1 s'}$ elements y in $\mathcal{HS}(z, r)$.

We say that a highly concentrated curve neighborhood $\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z)$ covers $\Omega'(z')$ ($z' \in \mathcal{I}_z$) if $\mathfrak{D}_z(z') \in \mathcal{C}$.

To get a contradiction to (11.12), it is enough to show that if [Assumption 11.10](#) holds, then the union

$$(11.14) \quad \bigcup_{y \in \mathcal{HS}(z, r)} \mathcal{C}(\mathcal{T}(y), z)$$

(which by definition corresponds to a subset of $\Theta'(z)$) must cover at least $e^{d_i s' + 20C_1 \varepsilon_1 s'}$ different full $\Omega'(z')$ with $z' \in \mathcal{I}_z$. Indeed, if this holds, then by the definition of full neighborhoods in (11.1), we have that

$$\tilde{\mu}(\Theta'(z)) \geq e^{-2s'} e^{-(d_i - 12C_1 \varepsilon_1)s'},$$

which contradicts (11.12).

Now we aim to show that, under the assumption [Assumption 11.10](#), the following holds:

$$(11.15) \quad \begin{aligned} & \bigcup_{y \in \mathcal{HS}(z, r)} \mathcal{C}(\mathcal{T}(y), z) \text{ covers at least } e^{d_i s' + 20C_1 \varepsilon_1 s'} \\ & \text{different full } \Omega'(z'), z' \in \mathcal{I}_z. \end{aligned}$$

Given a highly concentrated $\mathcal{C}$ with equation

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0,$$

let us denote $k(\mathcal{C}) := \max\{|w_1/v_1|, |v_1/w_1|\}$. By dyadic pigeonholing, we have that there is some ℓ for which there are at least $e^{2d_i s' - 16C_1 \varepsilon_1 s'}$ different $y \in \mathfrak{J} \cap \Sigma^r$ such that $\mathcal{C} = \mathcal{C}(\mathcal{T}(y), z)$ satisfies $k(\mathcal{C}) \in [e^\ell, e^{\ell+1}]$. We divide the proof into two cases.

Case 1. $\ell \geq s'$: By [Remark 8.16.\(2\)](#), in this case the highly concentrated curve neighborhoods are neighborhoods of straight lines. Without loss of generality, we can assume that $|w_1/v_1| \in [e^\ell, e^{\ell+1}]$, in which case these highly concentrated curve neighborhoods are neighborhoods of almost vertical straight lines. Let $\mathcal{C} = \mathcal{C}(\mathcal{T}(y_0), z)$ be a highly concentrated curve; the definition of highly concentrated curve implies that $\|(v_1, w_1)\| \geq e^{s' - 10C_1 \varepsilon_1 s'}$; hence [Lemma 11.5](#) can be applied. Thus there are at most $e^{4s' + 60C_1 \varepsilon_1 s'}$ different $y \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C}(\mathcal{T}(y), z) \approx \mathcal{C}.$$

From this fact, it follows that we can find a collection $\mathfrak{CT}$ of highly concentrated curve neighborhoods in $\Theta'(z)$ with at least

$$e^{2d_i s' - 16C_1 \varepsilon_1 s'} e^{-4s' - 60C_1 \varepsilon_1 s'} = e^{2d_i s' - 4s' - 76C_1 \varepsilon_1 s'}$$

elements such that any two curve neighborhoods from $\mathfrak{CT}$ are not comparable.

Now let us estimate the number of different full $\Omega'(z')$'s (with $z' \in \mathcal{I}_z$) covered by highly concentrated curve neighborhoods from $\mathfrak{CT}$. Our goal is to show that there are at least

$$e^{d_i s' + 20C_1 \varepsilon_1 s'}$$

such $\Omega'(z')$'s.

Note that every highly concentrated curve neighborhood must cover at least $e^{2s' - 9C_1 \varepsilon_1 s'}$ full $\Omega'(z')$'s. Thus, counting with multiplicity, the curve neighborhoods from $\mathfrak{CT}$ cover at least

$$e^{2d_i s' - 2s' - 85C_1 \varepsilon_1 s'}$$

full $\Omega'(z') \subset \Theta'(z)$. If every full $\Omega'(z')$ is counted with multiplicity at most

$$e^{d_i s' - 2s' - 105C_1 \varepsilon_1 s'},$$

then we are done.

Therefore assume the following:

Assumption 11.11. There exists a full $\Omega'(z_1) \subset \Theta'(z)$ which is counted with multiplicity not fewer than $e^{d_i s' - 2s' - 105C_1 \varepsilon_1 s'}$; let $(X_0, Y_0) = \mathfrak{D}_z(z_1)$.

Let $\mathfrak{L}\mathfrak{L} \subset \mathfrak{C}\mathfrak{T}$ denote the set of highly concentrated curve neighborhoods passing through (X_0, Y_0) . Then

$$|\mathfrak{L}\mathfrak{L}| \geq e^{d_i s' - 2s' - 105C_1 \varepsilon_1 s'}.$$

Recalling that in the case we are currently considering, namely of $\ell \geq s'$, these highly concentrated curve neighborhoods are neighborhoods of straight lines, as well as the definition of comparable in [Definition 11.3](#), we have that any two non-comparable curve neighborhoods $\mathcal{C}$ and $\mathcal{C}'$ passing through (X_0, Y_0) do not intersect at any (X, Y) with $|Y - Y_0| \geq e^{-s' - 11C_1 \varepsilon_1 s'}$. Thus, the highly concentrated curve neighborhoods from $\mathfrak{L}\mathfrak{L}$ cover at least

$$e^{d_i s' - 120C_1 \varepsilon_1 s'}$$

different full $\Omega'(z'')$'s. Moreover, all these highly concentrated curve neighborhoods are contained in

$$\mathcal{N}_1^{s'} = \mathcal{N}^{s'}(X_0, Y_0) = (X_0, Y_0) + [\beta e^{-2s'}] \times [\beta e^{-s'}].$$

Similarly to the definition of $\mathcal{HS}(z, r)$ and $\mathcal{SS}(z, r)$ (cf. [\(9.7\)](#) and [\(9.8\)](#)), let us define

$$(11.16) \quad \begin{aligned} & \mathcal{HS}(z, r; \mathcal{N}_1^{s'}) \\ & := \left\{ y \in \mathfrak{J} \cap \Sigma^r : \mathcal{C}(\mathcal{T}(y), z) \subset \mathcal{N}_1^{s'} \text{ and is highly concentrated} \right\}, \end{aligned}$$

and

$$(11.17) \quad \mathcal{SS}(z, r; \mathcal{N}_1^{s'}) := \bigcup_{y \in \mathcal{HS}(z, r; \mathcal{N}_1^{s'})} \Omega'(y).$$

We claim that

$$(11.18) \quad \tilde{\mu}(\mathcal{SS}(z, r; \mathcal{N}_1^{s'})) \leq e^{-2s'} e^{-800C_1 \varepsilon_1 s'}.$$

Indeed, by [Lemma 11.8](#), there exists $z' \in \Theta'(z)$ such that the following holds: for any $y \in \mathcal{HS}(z, r; \mathcal{N}_1^{s'})$, we have

$$\Omega'(y) \subset a(2s')Q(\mathfrak{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z},$$

where $\tilde{z} = a(-2s')z'$. Therefore,

$$\mathcal{SS}(z, r; \mathcal{N}_1^{s'}) \subset \left(a(2s')Q(\mathfrak{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z} \right) \cap \Sigma^r.$$

By our discussion at the beginning of [Section 7](#), [Proposition 6.3](#) holds for $\mu_{\mathcal{F}_i}$. Thus, we have

$$\begin{aligned} \tilde{\mu} \left(a(2s')Q(\mathfrak{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z} \right) & \leq \beta^{-15} \mu_{\mathcal{F}_i} \left(Q(\mathfrak{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z} \right) \\ & \leq \beta^{-15} e^{-d_0(s'-1)/2} < e^{-800C_1 \varepsilon_1 s'}. \end{aligned}$$

Note that

$$\left(a(2s')Q(\mathfrak{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z} \right) \cap \Sigma^r$$

is a cross-section of $a(2s')Q(\mathbf{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z}$ along the U -direction. It follows that

$$\begin{aligned}\tilde{\mu}\left(\mathcal{SS}(z, r; \mathcal{N}_1^{s'})\right) &\leq \tilde{\mu}\left(\left(a(2s')Q(\mathbf{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z}\right) \cap \Sigma^r\right) \\ &\leq e^{-2s'}\tilde{\mu}\left(a(2s')Q(\mathbf{r}_1, s' - 1, \beta)B^{A_0H}(\beta)\tilde{z}\right) \\ &\leq e^{-2s'}e^{-800C_1\varepsilon_1s'}.\end{aligned}$$

This proves (11.18). Thus, [Assumption 11.10](#) holds with $\mathcal{HS}(z, r)$ replaced by

$$\mathcal{HS}(z, r) \setminus \mathcal{HS}(z, r; \mathcal{N}_1^{s'}).$$

By repeating the above argument with $\mathcal{HS}(z, r)$ replaced by

$$\mathcal{HS}(z, r) \setminus \mathcal{HS}(z, r; \mathcal{N}_1^{s'}),$$

we will get $\mathcal{N}_2^{s'}$ disjoint with $\mathcal{N}_1^{s'}$, and at least $e^{d_i s' - 120C_1\varepsilon_1 s'}$ different $\Omega(z'')$'s with $\mathfrak{D}_z(z'') \in \mathcal{N}_2^{s'}$. This argument can be repeated for at least $e^{400C_1\varepsilon_1 s'}$ times. This gives at least

$$e^{d_i s' - 120C_1\varepsilon_1 s'} e^{400C_1\varepsilon_1 s'} \geq e^{d_i s' + 20C_1\varepsilon_1 s'}$$

different full $\Omega'(z')$'s. This establishes (11.15), completing the proof for the case $\ell \geq s'$.

Case 2. $\ell < s'$: By [Lemma 11.5](#), we have that for every highly concentrated curve neighborhood $\mathcal{C}$, there exist at most

$$e^{2s' + 2\ell + 60C_1\varepsilon_1 s'}$$

different $y \in \mathfrak{J} \cap \Sigma^r$ such that

$$\mathcal{C}(\mathcal{T}(y), z) \approx \mathcal{C}.$$

Therefore, we can choose a collection $\mathfrak{CT}$ of highly concentrated curve neighborhoods in $\Theta'(z)$ such that

$$|\mathfrak{CT}| \geq e^{2d_i s' - 16C_1\varepsilon_1 s'} e^{-2s' - 2\ell - 60C_1\varepsilon_1 s'} = e^{(2d_i - 2)s' - 2\ell - 76C_1\varepsilon_1 s'},$$

and any two elements from $\mathfrak{CT}$ are not comparable. Note that every highly concentrated curve neighborhood covers at least

$$e^{2s' - 9C_1\varepsilon_1 s'}$$

different full $\Omega'(z')$'s. Thus, counting multiplicity, highly concentrated curve neighborhoods from $\mathfrak{CT}$ cover at least

$$e^{2d_i s' - 2\ell - 85C_1\varepsilon_1 s'}$$

full $\Omega'(z')$'s.

Our goal is to prove the following: The curve neighborhoods from $\mathfrak{CT}$ cover at least $e^{d_i s' + 20C_1\varepsilon_1 s'}$ different full $\Omega'(z')$.

If every full $\Omega'(z')$ is covered by at most

$$e^{d_i s' - 2\ell - 105C_1 \varepsilon_1 s'}$$

different highly concentrated curve neighborhoods, then we are done. Now let us assume the following:

Assumption 11.12. there exists a full $\Omega'(z_0)$ that is covered by at least $e^{d_i s' - 2\ell - 105C_1 \varepsilon_1 s'}$ different highly concentrated curve neighborhoods.

We write $(X_0, Y_0) = \mathfrak{D}_z(z_0)$.

Let $\mathfrak{L}\mathfrak{L} \subset \mathfrak{C}\mathfrak{T}$ denote the set of highly concentrated curve neighborhoods passing through (X_0, Y_0) .

Now let us consider the following process:

Step 0: We start with $\mathfrak{L}_0 = \emptyset$ and $\mathfrak{P}_0 = \emptyset$.

Step $j+1$: Suppose $\mathfrak{L}_j$ and $\mathfrak{P}_j$ have been constructed in Step j . We pick a highly concentrated curve neighborhood $\mathcal{C} \in \mathfrak{L}\mathfrak{L}$ such that the number of full $\Omega'(z')$ contained in $\mathcal{C}$ but not contained in $\mathfrak{P}_j$ is at least $\frac{1}{2}e^{2s' - 9C_1 \varepsilon_1 s'}$. Then we define

$$\mathfrak{L}_{j+1} = \mathfrak{L}_j \cup \{\mathcal{C}\}, \text{ and } \mathfrak{P}_{j+1} = \mathfrak{P}_j \cup \mathcal{C}.$$

If we can not pick such a curve neighborhood, the process stops.

When the process stops, say at step n , let us take any $\mathcal{C} \in \mathfrak{L}\mathfrak{L} \setminus \mathfrak{L}_n$. Then, according to the way the $\mathfrak{L}_j$ and $\mathfrak{P}_j$ were constructed, we have that at least

$$\frac{1}{2}e^{2s' - 9C_1 \varepsilon_1 s'}$$

full $\Omega'(z')$ in $\mathcal{C}$ are covered by curve neighborhoods from $\mathfrak{L}_n$. Let the equation of $\mathcal{C}$ be

$$(X - X_0)(Y - Y_0) + v_1^{-1}(X - X_0) + w_1^{-1}(Y - Y_0) = 0.$$

Using dyadic pigeonholing, we can assume that there exists $m_{\mathcal{C}} > 0$, such that at least

$$e^{2s' - 9.1C_1 \varepsilon_1 s'}$$

full $\Omega'(z')$ in $\mathcal{C}$ are covered by curve neighborhoods from $\mathfrak{L}_n$ whose equation,

$$(X - X_0)(Y - Y_0) + v_1'^{-1}(X - X_0) + w_1'^{-1}(Y - Y_0) = 0,$$

satisfies the following:

$$(11.19) \quad |w_1 - w_1'| \in [e^{s' - m_{\mathcal{C}}}, e^{s' - m_{\mathcal{C}} + 1}].$$

We take $m_{\mathcal{C}}$ to be the minimal natural number satisfying this condition. Let us denote the collection of the curve neighborhoods satisfying (11.19) by $\mathfrak{L}_n(\mathcal{C}, m_{\mathcal{C}})$. We also remove $\mathcal{C}'$ from $\mathfrak{L}_n(\mathcal{C}, m_{\mathcal{C}})$ if the intersection $\mathcal{C}' \cap \mathcal{C}$ is contained in the $e^{-s' - 10C_1 \varepsilon_1 s'}$ -neighborhood of (X_0, Y_0) . By Lemma 11.6, we have that the curve neighborhoods from $\mathfrak{L}_n(\mathcal{C}, m_{\mathcal{C}})$ are contained in the

$e^{-s'-m_C-2\ell}$ -neighborhood of $\mathcal{C}$. Let us call this curve neighborhood the (ℓ, m_C) -neighborhood of $\mathcal{C}$. By the last statement of Lemma 11.7, we have that each curve neighborhood from $\mathfrak{L}_n(\mathcal{C}, m_C)$ can cover at most

$$e^{m_C+2\ell+11C_1\varepsilon_1s'}$$

different full $\Omega'(z')$ contained in $\mathcal{C}$. Therefore, we have that

$$|\mathfrak{L}_n(\mathcal{C}, m_C)| \geq e^{2s'-m_C-2\ell-21C_1\varepsilon_1s'}.$$

By the construction of $\mathfrak{L}_n$, every $\mathcal{C}' \in \mathfrak{L}_n(\mathcal{C}, m_C)$ covers at least

$$e^{2s'-10C_1\varepsilon_1s'}$$

different full $\Omega'(z')$ that are not covered by any other $\mathcal{C}'' \in \mathfrak{L}_n(\mathcal{C}, m_C)$. Thus, curve neighborhoods from $\mathfrak{L}_n(\mathcal{C}, m_C)$ cover at least

$$(11.20) \quad e^{4s'-m_C-2\ell-31C_1\varepsilon_1s'}$$

different full $\Omega'(z')$ in the (ℓ, m_C) -neighborhood of $\mathcal{C}$. We say that a (ℓ, m') -neighborhood is η -almost filled if it covers at least $\eta e^{4s'-m'-2\ell}$ different full $\Omega'(z')$. By repeating the above argument for every curve neighborhood $\mathcal{C} \in \mathfrak{L}\mathfrak{L} \setminus \mathfrak{L}_n$, we have that the (ℓ, m_C) -neighborhood of every curve neighborhood $\mathcal{C} \in \mathfrak{L}\mathfrak{L} \setminus \mathfrak{L}_n$ is $e^{-31C_1\varepsilon_1s'}$ -almost filled.

By dyadic pigeonholing, there exists a $m \in \mathbb{N}$ such that $m = m_C$ for at least β -proportion of the curve neighborhoods $\mathcal{C} \in \mathfrak{L}\mathfrak{L}$. Let $\tilde{\mathfrak{N}}$ denote the collection of the (ℓ, m) -neighborhoods obtained as above.

Given a (ℓ, m) -neighborhood $\tilde{\mathcal{C}}$, let us denote by $L(\tilde{\mathcal{C}})$ the

$$e^{-s'-m-2\ell+C_1\varepsilon_1s'}$$

neighborhood of $\tilde{\mathcal{C}}$ and call it the *enlarged (ℓ, m) -neighborhood* of $\tilde{\mathcal{C}}$. Note that if two (ℓ, m) -neighborhoods $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ both contain a curve neighborhood $\mathcal{C}$ from $\mathfrak{L}_n$, then we have

$$\tilde{\mathcal{C}}_1 \subset L(\tilde{\mathcal{C}}_2).$$

Let us select a subset $\mathfrak{N} \subset \tilde{\mathfrak{N}}$ satisfying the following:

- For any $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2 \in \mathfrak{N}$,

$$\tilde{\mathcal{C}}_1 \not\subset L(\tilde{\mathcal{C}}_2).$$

- For any $\tilde{\mathcal{C}} \in \tilde{\mathfrak{N}} \setminus \mathfrak{N}$, there exists $\tilde{\mathcal{C}}' \in \mathfrak{N}$ such that

$$\tilde{\mathcal{C}} \subset L(\tilde{\mathcal{C}}').$$

Let us consider the curve neighborhoods from $\mathfrak{L}_n$ which are contained in some (ℓ, m) -neighborhood from $\mathfrak{N}$. We have that those curve neighborhoods cover at least

$$|\mathfrak{N}| e^{4s'-m-2\ell-31C_1\varepsilon_1s'}$$

different full $\Omega(z')$. If

$$|\mathfrak{N}| \geq e^{d_1s'-4s'+m+2\ell+51C_1\varepsilon_1s'},$$

then we are done as this establishes the goal above. Now suppose that

$$(11.21) \quad |\mathfrak{N}| < e^{d_i s' - 4s' + m + 2\ell + 51C_1 \varepsilon_1 s'}.$$

On the other hand, by the definition of $\mathfrak{N}$, we have that at least β -proportional curve neighborhoods from $\mathfrak{L}\mathfrak{L}$ are contained in some enlarged (ℓ, m) -neighborhood $L(\tilde{\mathcal{C}})$, where $\tilde{\mathcal{C}} \in \mathfrak{N}$. By [Lemma 11.7](#), every enlarged (ℓ, m) -neighborhood contains at most

$$e^{4s' - 2m - 4\ell + 102C_1 \varepsilon_1 s'}$$

different curve neighborhoods from $\mathfrak{L}\mathfrak{L}$. We have

$$|\mathfrak{N}| e^{4s' - 2m - 4\ell + 102C_1 \varepsilon_1 s'} \geq e^{d_i s' - 2\ell - 105C_1 \varepsilon_1 s'},$$

which implies that

$$|\mathfrak{N}| \geq e^{d_i s' - 4s' + 2m + 2\ell - 207C_1 \varepsilon_1 s'}.$$

By [\(11.21\)](#), we get

$$m \leq 260C_1 \varepsilon_1 s'.$$

Then by [\(11.20\)](#), every (ℓ, m) -neighborhood contains at least

$$e^{4s' - 2\ell - 300C_1 \varepsilon_1 s'}$$

different full $\Omega'(z')$. If $\ell < 10^{10}C_1 \varepsilon_1 s'$, then it covers at least

$$e^{4s' - 3 \times 10^{10}C_1 \varepsilon_1 s'} \geq e^{d_i s' + 20C_1 \varepsilon_1 s'}$$

different full $\Omega'(z')$, and we are done.

Now we assume that $\ell \geq 10^{10}C_1 \varepsilon_1 s'$. This case is similar to the conclusion of [Case 1](#) ($\ell \geq s'$), and to deal with this case we will need to use [Proposition 6.3](#) (which roughly says that our Diophantine assumptions guarantee that $\tilde{\mu}$ does not concentrate along orbits of $\mathrm{SL}(2, R) \ltimes \mathbb{R}^2$).

According to our assumption, the (ℓ, m) -neighborhoods from $\mathfrak{N}$ cover at least

$$e^{d_i s' - 4s' + 2\ell - 207C_1 \varepsilon_1 s'} e^{4s' - 2\ell - 300C_1 \varepsilon_1 s'} = e^{d_i s' - 507C_1 \varepsilon_1 s'}$$

different full $\Omega'(z')$. Similarly to [Case 1](#), where $\ell \geq s'$, we have that all $\Omega'(z')$ covered by these (ℓ, m) -neighborhoods are contained in

$$\mathcal{N}_1^\ell = \mathcal{N}^\ell(X_0, Y_0) = (X_0, Y_0) + [\beta e^{-s' - \ell}] \times [\beta e^{-s'}].$$

As above, define

$$\mathcal{HS}(z, r; \mathcal{N}_1^\ell) := \left\{ y \in \mathfrak{J} \cap \Sigma^r : \mathcal{C}(\mathcal{T}(y), z) \subset \mathcal{N}_1^\ell \text{ and is highly concentrated} \right\}$$

and

$$\mathcal{SS}(z, r; \mathcal{N}_1^\ell) := \bigcup_{y \in \mathcal{HS}(z, r; \mathcal{N}_1^\ell)} \Omega'(y).$$

We claim that

$$\tilde{\mu}(\mathcal{SS}(z, r; \mathcal{N}_1^\ell)) \leq e^{-2s' - 800C_1 \varepsilon_1 s'}.$$

In fact, by [Lemma 11.8](#), there exists $z' \in \Theta'(z)$ such that the following holds: For any $y \in \mathfrak{J} \cap \Sigma^r$ such that $\mathcal{C}(\mathcal{T}(y), z) \subset \mathcal{N}_1^\ell$ and is highly concentrated, we have

$$\Omega'(y) \subset a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z},$$

where $\tilde{z} = a(-2s')z'$. Therefore,

$$\mathcal{SS}(z, r; \mathcal{N}_1^\ell) \subset a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z} \cap \Sigma^r.$$

By our discussion at the beginning of [Section 7](#), [Proposition 6.3](#) holds for $\mu_{\mathcal{F}_i}$. Thus, we have

$$\begin{aligned} \tilde{\mu}\left(a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z}\right) &\leq \beta^{-15}\mu_{\mathcal{F}_i}\left(Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z}\right) \\ &\leq \beta^{-15}e^{-d_0(\ell-1)/2} < e^{-800C_1\varepsilon_1s'}. \end{aligned}$$

Note that

$$a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z} \cap \Sigma^r$$

is a cross-section of $a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z}$ along the U -direction. We have that

$$\begin{aligned} \tilde{\mu}\left(\mathcal{SS}(z, r; \mathcal{N}_1^\ell)\right) &\leq \tilde{\mu}\left(a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z} \cap \Sigma^r\right) \\ &\leq e^{-2s'}\tilde{\mu}\left(a(2s')Q(\mathfrak{r}_1, \ell - 1, \beta)B^{A_0H}(\beta)\tilde{z}\right) \leq e^{-2s'-800C_1\varepsilon_1s'}. \end{aligned}$$

This proves the claim. Similarly to the case $\ell \geq s'$, the claim implies that [Assumption 11.10](#) holds with $\mathcal{HS}(z, r)$ replaced by $\mathcal{HS}(z, r) \setminus \mathcal{HS}(z, r; \mathcal{N}_1^\ell)$. By repeating the above argument with $\mathcal{HS}(z, r)$ replaced by $\mathcal{HS}(z, r) \setminus \mathcal{HS}(z, r; \mathcal{N}_1^\ell)$, we get another

$$\mathcal{N}_2^\ell = \mathcal{N}^\ell(X_2, Y_2) = (X_2, Y_2) + [\beta e^{-s'-\ell}] \times [\beta e^{-s'}]$$

disjoint with $\mathcal{N}_1^\ell$ that also contains at least

$$e^{d_i s' - 507C_1\varepsilon_1 s'}$$

different full $\Omega'(z')$. By our claim, the argument can be repeated for at least $e^{600C_1\varepsilon_1 s'}$ times, which gives at least

$$e^{d_i s' - 507C_1\varepsilon_1 s'} e^{600C_1\varepsilon_1 s'} \geq e^{d_i s' + 20C_1\varepsilon_1 s'}$$

different full $\Omega'(z')$.

This completes the proof. $\square$

12. Proof of the main theorem

In order to apply [Proposition 3.1](#), we first prove the following statement:

PROPOSITION 12.1. *Let $s \leq \hat{s} < t$, $s' = (t - \hat{s})/2$, and $\hat{\mu}$ be a probability measure on $X_{\beta^{1/4}}$ defined by the normalized Lebesgue measure on the U -orbit $a(\hat{s})u([1])x_0$ removed an exponentially small proportion. Suppose $\hat{\mu}$ is $(4-\theta, s')$ -good. Then $a(2s')_*\hat{\mu}$ can be written as*

$$(12.1) \quad a(2s')_*\hat{\mu} = \nu_{\S}\mu_{\S} + \sum_k \nu_k a(2s')_*\mu_k,$$

where $\mu_{\S}$ is a probability measure on X , $\nu_{\S} = O(\beta)$, and for each k , μ_k is a probability measure on

$$\exp(B_{\mathbf{r}+}(\beta))u([\beta])x_k$$

for some $x_k \in X$. Moreover, for any $f \in C_c^\infty(X)$,

$$\int f d\mu_k = \int_{[\beta]^2} \int_{[1]} f(\exp(v_1 \mathbf{v}_1 + w_1 \mathbf{w}_1)u(\beta r)) dr d\rho_k(v_1, w_1),$$

where ρ_k is a probability measure on $[\beta]^2$ with dimension larger than $2 - 2\theta$ at scale s' (see [Proposition 3.1](#) for the definition).

Proof. Let us define

$$(12.2) \quad \Upsilon_{s'} := B^{A_0 A U^*}(\beta) \exp(B_{\mathbf{r}-}(\beta e^{-s'})) \exp(B_{\mathbf{r}+}(\beta)) B^U(\beta),$$

and let us decompose $X_{\beta^{1/4}}$ as

$$X_{\beta^{1/4}} = \bigcup_{k=1}^{\beta^{-100}e^{2s'}} \Upsilon_{s'} x_k.$$

We say that x_k is *light* if

$$\hat{\mu}(\Upsilon_{s'} x_k) \leq \beta^{-2} e^{-2s' - \theta s'}.$$

Otherwise we say that x_k is *heavy*. For each heavy x_k , let us denote

$$\bar{\nu}_k = \hat{\mu}(\Upsilon_{s'} x_k)$$

and

$$\bar{\mu}_k = \bar{\nu}_k^{-1} \hat{\mu}|_{\Upsilon_{s'} x_k}.$$

Let us define

$$\Upsilon_{\text{light}} := \bigcup_{x_k \text{ light}} \Upsilon_{s'} x_k.$$

Then it is easy to see that

$$\hat{\mu}(\Upsilon_{\text{light}}) \leq \beta^{10}.$$

Therefore, we have

$$(12.3) \quad \hat{\mu} = \sum_{x_k \text{ heavy}} \bar{\nu}_k \bar{\mu}_k + O(\beta^{10}).$$

We claim that, for every heavy x_k , $\bar{\mu}_k$ is *nice*; namely, for $y \in \text{supp} \bar{\mu}_k$,

$$\bar{\mu}_k \left(Q_{s'}(\beta) B^{A_0 H}(\beta) y \right) \leq \beta^2 e^{-2s' + 2\theta s'}.$$

In fact, since $\hat{\mu}$ is $(4 - \theta, s')$ -good, we have that

$$\hat{\mu} \left(Q_{s'}(\beta) B^{A_0 H}(\beta) y \right) \leq e^{-s'(4-\theta)}.$$

Therefore,

$$\begin{aligned} \bar{\mu}_k \left(Q_{s'}(\beta) B^{A_0 H}(\beta) y \right) &= \hat{\mu}^{-1}(\Upsilon_{s'} x_k) \hat{\mu} \left(Q_{s'}(\beta) B^{A_0 H}(\beta) y \right) \\ &\leq \beta^2 e^{2s' + \theta s'} e^{-s'(4-\theta)} = \beta^2 e^{-2s' + 2\theta s'}. \end{aligned}$$

This proves the claim.

Now for every heavy x_k , let us define μ_k as follows: For each U -orbit $\mathcal{U}$ of length β contained in $\text{supp} \bar{\mu}_k$, pick $y \in \mathcal{U}$ such that

$$y \in B^{A_0 A U^*}(\beta) \exp \left(B_{\mathbf{r}^-}(\beta e^{-s'}) \right) \exp \left(B_{\mathbf{r}^+}(\beta) \right) x_k.$$

Then we can write

$$y = a_0 a u^* \exp(\mathbf{w}^-) \exp(\mathbf{w}^+) x_k,$$

where $a_0 a u^* \in B^{A_0 A U^*}(\beta)$, $\mathbf{w}^- \in B_{\mathbf{r}^-}(\beta e^{-s'})$, and $\mathbf{w}^+ \in B_{\mathbf{r}^+}(\beta)$. Let us define

$$\bar{y} := \exp(\mathbf{w}^+) x_k.$$

Since $\|\mathbf{w}^-\| \leq \beta e^{-s'}$, we have that

$$(12.4) \quad \bar{y} \in Q_{s'}(\beta) B^{A_0 H}(\beta) y.$$

Moreover, we may apply (8.7), (8.8), and (8.9) to conclude that for every $r' \in [\beta]$, there exists $f(r')$ such that $|r' - f(r')| \leq \beta |r'|$, and

$$a(2s') u(f(r')) \bar{y} \in B^G(\beta) u(r') y.$$

Therefore, for any function $f \in C_c^\infty(X)$,

$$(12.5) \quad \left| \int_{[1]} f(a(2s') u(\beta r) y) dr - \int_{[1]} f(a(2s') u(\beta r) \bar{y}) dr \right| \ll \beta \|f\|_{C^1}.$$

Let us denote the union of $\bar{y}$'s constructed as above by $\mathcal{T}$ and let us define μ_k as the normalized Lebesgue measure on the union

$$\bigcup_{\bar{y} \in \mathcal{T}} u([\beta]) \bar{y}.$$

By (12.5), we have that $a(2s')_*\mu_k$ and $a(2s')_*\bar{\mu}_k$ are $O(\beta)$ -close; namely, for any $f \in C_c^\infty(X)$,

$$\left| \int f d a(2s')_*\mu_k - \int f d a(2s')_*\bar{\mu}_k \right| \ll \beta \|f\|_{C^1}.$$

Since μ_k is supported on $\exp(B_{\mathbf{r}^+}(\beta))u([\beta])x_k$, we have that there exists a probability measure ρ_k on $[\beta]^2$ such that for $f \in C_c^\infty(X)$,

$$\int f d\mu_k = \int_{[\beta]^2} \int_{[1]} f(\exp(v_1\mathbf{v}_1 + w_1\mathbf{w}_1)u(\beta r)) d\rho_k(v_1, w_1) dr.$$

We claim that ρ_k has dimension larger than $2 - 2\theta$ at scale s' . In fact, by (12.4), we have that for y and $\bar{y}$ constructed as above,

$$u([\beta])\bar{y} \in Q_{s'}(\beta)B^{A_0H}(\beta)y.$$

Therefore, given $\bar{\mu}_k$ is nice, we have μ_k is also nice. Therefore, for any square S of size $\beta e^{-s'}$, we have

$$\rho_k(S) = \mu_k(Q_{s'}(\beta)B^{A_0H}(\beta)y_S) \leq \beta^2 e^{-2s' + 2\theta s'}.$$

Here $y_S = \exp(v_S\mathbf{v}_1 + w_S\mathbf{w}_1)$, where (v_S, w_S) is the center of S . This proves the claim.

Combining the claim above with (12.3) and (12.5), we obtain (12.1). $\square$

We are now equipped to prove Theorem 1.3.

Proof of Theorem 1.3. By Proposition 6.2, we have that $\mu_{\mathcal{F}_0}$ is (d_0, s_0) -good. By Propositions 9.9, 9.10, and 11.1, we get Proposition 9.1. By Proposition 7.2, it implies that $\mu_{\mathcal{F}_i}$ is (d_i, s_i) -good, where $d_i = d_0 + i\varepsilon_2$ and $s_i = 2^{-i}s_0$ (we remove an exponentially small proportion at each step). We can proceed with this inductive construction until $d_i \geq 4 - \theta$. By Proposition 12.1, we have that

$$a(2s_i)_*\mu_{\mathcal{F}_i} = \sum_k \nu_k a(2s_i)_*\mu_k + O(\beta).$$

Note that by Proposition 12.1, every μ_k is defined by a measure ρ_k on $[\beta]^2$ with dimension larger than $2 - 2\theta$ at scale s' . Then by Proposition 3.1, we have that $a(2s_i)_*\mu_k$ is effectively equidistributed. This implies that $a(2s_i)_*\mu_{\mathcal{F}_i}$ is effectively equidistributed. Noting that $\mu_{\mathcal{F}_i}$ differs from the normalized Lebesgue measure on

$$a(t - 2s_i)u([1])x_0$$

only by removing an exponentially small proportion, we complete the proof. $\square$

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